On the Limits to Invariance in the Twist/Wrench and Motor Representations of Motion and Force Vectors

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Abstract

This paper examines a property of twists and wrenches that leads to a restriction in the choice of basis vectors in a twist/wrench representation of motion and force vectors. Given any non-zero twist $\mathbf{t}$ (or wrench $\mathbf{w}$), it is possible to identify wrenches $\mathbf{w}$ (or twists $\mathbf{t}$) having the same screw axis as $\mathbf{t}$ (or $\mathbf{w}$). This kind of relationship is inherent in the definition of twists and wrenches, and should therefore be regarded as an invariant property; but it is not invariant with respect to a general change of basis, so it becomes necessary to restrict the choice of bases to a set that does preserve the invariance of this property. A similar problem arises with motors.

This paper goes on to argue that the restricted choice of bases is a practical disadvantage on the grounds that it hinders the analysis of freedoms and constraints, and reduces the number of analytical techniques that can be used.

1 Introduction

In screw theory, a twist is a vectorial quantity that describes the infinitesimal motion of a rigid body. Likewise, a wrench is a vectorial quantity that describes the resultant of a system of forces acting on a rigid body. Both are defined in terms of a magnitude and a screw axis. A twist is defined as a displacement of a specific (infinitesimal) amplitude about a specific screw axis, and similarly for a wrench. A screw axis is, in turn, defined as a directed line in 3-D space together with a scalar that specifies its pitch [1].

The screw axis is therefore a property that is common to both twists and wrenches; so it is possible to define a relation between twist space and wrench space along the following lines: given any nonzero twist $\mathbf{t}$ (or wrench $\mathbf{w}$), it is possible to identify an infinite number of wrenches $\mathbf{w}$ (or twists $\mathbf{t}$), all of which have the same screw axis as the given twist $\mathbf{t}$ (or wrench $\mathbf{w}$). Furthermore, the screw axis is an intrinsic property of twists and wrenches, so this common-screw relation should be invariant with respect to the choice of basis vectors in a coordinate-based representation.

If we wish to represent twists and wrenches by means of coordinate vectors, then we must choose a representation where their invariant properties are preserved under any allowable change of basis. Unfortunately, the common-screw relation is not invariant with respect to every possible change of basis, so it becomes necessary to limit the choices to a subset within which the invariant properties are preserved.

A similar problem arises in the motor calculus, where twists and wrenches are all expressed using motors [9]. In this case, there is an invariant common-motor relation between twists and wrenches, which subsumes the common-screw relation.

On the other hand, there is no such problem in a dual-space representation of motions and forces, where the two types of vector are kept in two dual, but otherwise completely unrelated, vector spaces. In this representation, only the invariance of the scalar product must be preserved.

This paper investigates the invariance problem caused by the presence of the common-screw and common-motor relations. It shows that the most commonly-used systems of coordinates do indeed preserve the relevant invariants; but it also shows that there are many simple changes of basis that do not, including simple scalings and reflections.

This paper goes on to argue that the restricted choice of basis vectors is a genuine practical disadvantage, despite the fact that the most commonly-used choices are all in the allowed set, because the set is too small to support a new analytical tool that simplifies the analysis of freedoms and constraints by means of a general change of basis [6]. The analyst who chooses the twist/wrench or motor representations must do without this tool.

The rest of this paper is organized as follows. First, the dual-space representation is described, along with the coordinate transformation rules that preserve the invariance of the scalar product. The next two sections explore the problem of preserving the invariance of the common-motor and common-screw relations while simultaneously preserving the invariance of the scalar product. The investigations are carried out by simulating the relevant properties of the motor and twist/wrench representations within the dual-space representation. The final section examines the consequences, and puts the argument in favour of preferring the dual-space representation for analysing the dynamics and kinestatics of rigid-body systems.

2 The Dual-Space Representation

Let M^n and F^n be two n-dimensional vector spaces with the property that a scalar product is defined between them. If $\mathbf{m} \in \mathsf{M}^n$ and $\mathbf{f} \in \mathsf{F}^n$ then the scalar product may be written either $\mathbf{m} \cdot \mathbf{f}$ or $\mathbf{f} \cdot \mathbf{m}$, both meaning the same thing. Neither space has an inner product, so the expressions $\mathbf{m} \cdot \mathbf{m}$ and $\mathbf{f} \cdot \mathbf{f}$ are not defined. Together, the two spaces and the scalar product define a dual-system of vector spaces, denoted $\langle \mathsf{M}^n, \mathsf{F}^n, \cdot \rangle$.

In the dual-space representation, M^n contains motion-type vectors and F^n contains force-type vectors. Examples of motion-type vectors include velocities, accelerations, infinitesimal displacements and joint motion axes. Examples of force-type vectors include forces, momenta and contact normals. Although we will be concerned mainly with vectors describing the motions of, and forces acting upon, a single rigid body, the elements of M^n and F^n are not tied to any particular physical interpretation. They could just as easily be vectors of generalized motions and forces for a complex multibody system. The only important property is that a scalar product be defined between them, such that its value can be interpreted as the work (power, etc.) done by a force acting over a small displacement (velocity, etc.).

In this paper, M^n and F^n are each used as umbrella terms for a collection of spaces that are formally distinct, but mathematically equivalent. For example, a statement like $\mathbf{v}, \mathbf{a} \in \mathsf{M}^n$, where $\mathbf{v}$ is a velocity vector and $\mathbf{a}$ is an acceleration, is intended as a short-hand notation for $\mathbf{v} \in \mathsf{M}^n_v$, $\mathbf{a} \in \mathsf{M}^n_a$, where M^n_v and M^n_a are spaces of velocity and acceleration vectors, respectively. Clearly, it is not possible for $\mathbf{v}$ and $\mathbf{a}$ to be literally members of the same space, since they have different physical units and $\mathbf{v} + \mathbf{a}$ is a meaningless expression; but the topic of discussion is the mathematical properties of various spaces, and M^n_v and M^n_a do have identical properties, so it is pointless to distinguish between them in the present context.

Starting with $\langle \mathsf{M}^n, \mathsf{F}^n, \cdot \rangle$, let us define a basis $D = \{\mathbf{d}_1, \ldots, \mathbf{d}_n\}$ on M^n and a basis $E = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ on F^n. If $\mathbf{m}_D$ and $\mathbf{f}_E$ are $n \times 1$ coordinate-vector representations of $\mathbf{m}$ and $\mathbf{f}$ in D and E then

$$\mathbf{m}_D = \begin{bmatrix} m_1 \\ \vdots \\ m_n \end{bmatrix} \quad \text{such that} \quad \mathbf{m} = \sum_{i=1}^{n} m_i \, \mathbf{d}_i \, ,$$

and similarly for $\mathbf{f}_E$. The scalar product is given by

$$\mathbf{m} \cdot \mathbf{f} = \mathbf{m}_D^T \, \mathbf{P}_{DE} \, \mathbf{f}_E \, ,$$

where

$$\mathbf{P}_{DE} = \begin{bmatrix} \mathbf{d}_1 \cdot \mathbf{e}_1 & \cdots & \mathbf{d}_1 \cdot \mathbf{e}_n \\ \vdots & & \vdots \\ \mathbf{d}_n \cdot \mathbf{e}_1 & \cdots & \mathbf{d}_n \cdot \mathbf{e}_n \end{bmatrix} .$$

(One of the defining properties of a dual system that wasn't mentioned earlier is that the scalar product must be such that $\mathbf{P}_{DE}$ is always nonsingular.)

As there are always two bases involved, it is convenient to introduce the idea of a *basis pair*, which is nothing more than a pair of bases, one on each vector space. If we say that A is a basis pair on $\langle \mathsf{M}^n, \mathsf{F}^n, \cdot \rangle$ then we are simply saying that A is an entity comprising two bases, one on M^n and one on F^n. We may call them A_M and A_F, respectively. The introduction of basis pairs allows us the following notational convenience: for any $\mathbf{m} \in \mathsf{M}^n$, $\mathbf{f} \in \mathsf{F}^n$ and basis pair A on $\langle \mathsf{M}^n, \mathsf{F}^n, \cdot \rangle$, $\mathbf{m}_A$ is the coordinate representation of $\mathbf{m}$ in A_M and $\mathbf{f}_A$ is the coordinate representation of $\mathbf{f}$ in A_F.

A basis pair can be defined simply by listing its constituents. For example, we may define A by $A = (D, E)$, in which case $A_\mathsf{M} = D$, $A_\mathsf{F} = E$, $\mathbf{m}_A = \mathbf{m}_D$ and $\mathbf{f}_A = \mathbf{f}_E$. We may also say that $\mathbf{P}_{DE} = \mathbf{P}_A$.

For any given choice of D, there is a unique basis E for which $\mathbf{P}_{DE}$ is the identity matrix. Let us call this basis $\mathrm{Recip}(D)$. The two bases D and $\mathrm{Recip}(D)$ are said to be reciprocal, and together they constitute a *reciprocal basis pair*. If A is any reciprocal basis pair on $\langle \mathsf{M}^n, \mathsf{F}^n, \cdot \rangle$ then $\mathbf{P}_A$ is the identity matrix, and

$$\mathbf{m} \cdot \mathbf{f} = \mathbf{m}_A^T \mathbf{f}_A \,.$$

The term 'reciprocal' is used here with a different meaning to the one it has in screw theory. In particular, reciprocity in screw theory is a relationship between two objects of the same type (two screws), but in the dual-space representation it is a relationship between objects of different type. This means that screw-theory concepts like self-reciprocity have no counterpart in the dual-space representation.

The reciprocal basis pair is the dual-space equivalent of an orthonormal basis in a Euclidean space: they both allow a scalar product $\mathbf{v} \cdot \mathbf{w}$ to be written as $\mathbf{v}^T \mathbf{w}$. However, there are n^2 freedoms available to choose a reciprocal basis pair on $\langle \mathsf{M}^n, \mathsf{F}^n, \cdot \rangle$, but only $n(n-1)/2$ freedoms to choose an orthonormal basis on Euclidean n-space. Differences like this help to illustrate that a dual system of vector spaces really is profoundly different from an inner-product space.

From here on, we restrict our choice of bases to reciprocal basis pairs only. This restriction guarantees the invariance of the scalar product, both in form and in value.

Let us now examine the rules governing a change of basis in $\langle \mathsf{M}^n, \mathsf{F}^n, \cdot \rangle$. Let $\mathbf{m}_A$, $\mathbf{f}_A$, $\mathbf{m}_B$ and $\mathbf{f}_B$ be representations of the two vectors $\mathbf{m} \in \mathsf{M}^n$ and $\mathbf{f} \in \mathsf{F}^n$, expressed in the reciprocal basis pairs A and B respectively. The formulas for performing a change of basis are:

$$\mathbf{m}_B = \mathbf{X}_m \, \mathbf{m}_A, \quad \mathbf{f}_B = \mathbf{X}_f \, \mathbf{f}_A,$$
$$\mathbf{m}_A = \mathbf{X}_m^{-1} \, \mathbf{m}_B, \quad \mathbf{f}_A = \mathbf{X}_f^{-1} \, \mathbf{f}_B,$$

where $\mathbf{X}_m$ is a coordinate transformation matrix that performs the change of basis in M^n, and $\mathbf{X}_f$ does the same in F^n. To preserve the invariance of the scalar product, $\mathbf{X}_m$ and $\mathbf{X}_f$

must satisfy the equation $\mathbf{m}_A^T \mathbf{f}_A = \mathbf{m}_B^T \mathbf{f}_B$ for all A and B, hence

$$\mathbf{X}_m^T \equiv \mathbf{X}_f^{-1}.$$

There is much more to be said about the dual-space representation, but the above is sufficient for the argument in this paper. More details can be found in [6].

Before moving on, let us examine two existing concepts that closely resemble the dual-space representation, but which actually behave like the motor representation with respect to the invariance properties under investigation.

The first is the idea of a dual system of coordinates: a single vector space is endowed with two types of coordinate system; typically, one is labelled covariant and the other contravariant, and the underlying bases are reciprocal to each other. A good example is the use of Plücker axis and ray coordinates to represent twists and wrenches, respectively, as done in [8, 10]. However, despite the segregation of twists and wrenches by coordinate system, there is still only a single underlying vector space; so twists and wrenches must ultimately be members of the same vector space, just as they are in the motor representation.

The second is the spatial algebra described in [4]. This algebra maintains a distinction between motion-type and force-type vectors that is sufficient to qualify it as a two-space representation; but the two spaces are deliberately designed to obey the same transformation rules under a change of basis, so that the two spaces can be treated as a single space for the purpose of defining transformation matrices, etc. For this reason, the relevant invariance properties of spatial algebra are the same as those of the motor representation. There is now a new version of spatial algebra, described briefly in the appendix of [5], which is based explicitly on the dual-space representation.

3 Invariance in the Motor Representation

This section investigates the restrictions that must be placed on the choice of bases when motion and force vectors are represented using motors.

Motors can be defined in various different ways, and can be regarded as geometrical objects in much the same way as screws. One convenient definition is to say that motors are screws with magnitudes. They are therefore six-parameter entities that are characterized by two numbers (magnitude and pitch) and a directed line in space. It can be shown that motors behave like vectors, and that the set of motors forms a vector space.

Motors can be given two different mathematical structures: a 6-D vector space over the real numbers [9], and a 3-D module over the ring of dual numbers [2]. We shall use the former, but similar results could be obtained using the latter.

For our purposes, the single most important property of the motor representation is that the same set of motors is used to represent both twists and wrenches. Since every twist can be represented by a motor, and each motor uniquely represents a single twist, the motor representation of twists involves a $1 : 1$ mapping between motors and twists. Similarly, the motor representation of wrenches involves a $1 : 1$ mapping between motors and wrenches.

These two mappings imply a $1 : 1$ relation between twists and wrenches: the common-motor relation. This relation is more specific than the common-screw relation, and subsumes it in the sense that a common motor automatically implies a common screw axis. It can be shown that the common-motor relation is linear.

The common-motor relation is a consequence of the act of representing twists and wrenches using motors. It can be defined in various geometrical ways that do not involve coordinates; so it should be regarded as an invariant relation within the motor representation.

To investigate the effect of the common-motor relation, let us model the motor representation by means of the dual system $\langle \mathsf{M}^6, \mathsf{F}^6, \cdot \rangle$ and the linear mapping $\boldsymbol{\Delta} : \mathsf{F}^6 \mapsto \mathsf{M}^6$, where M^6 is twist space, F^6 is wrench space, '$\cdot$' evaluates to the scalar product of two motors, and $\boldsymbol{\Delta}$ maps each wrench to the twist represented by the same motor. $\boldsymbol{\Delta}$ itself is a nonsingular 6×6 matrix.

Let $\mathbf{m}_A$, $\mathbf{m}_B$, $\mathbf{f}_A$ and $\mathbf{f}_B$ be coordinate vectors representing $\mathbf{m} \in \mathsf{M}^6$ and $\mathbf{f} \in \mathsf{F}^6$ in the reciprocal basis pairs A and B. Reciprocal basis pairs automatically preserve the invariance of the scalar product, so we can concentrate on finding the conditions under which $\boldsymbol{\Delta}$ is invariant. For a change of basis from A to B, the invariance condition is that

$$\mathbf{m}_A = \boldsymbol{\Delta}\,\mathbf{f}_A \quad \Rightarrow \quad \mathbf{m}_B = \boldsymbol{\Delta}\,\mathbf{f}_B\,.$$

This can only be true if

$$\boldsymbol{\Delta} = \mathbf{X}_m\,\boldsymbol{\Delta}\,\mathbf{X}_m^T\,, \tag{1}$$

where $\mathbf{X}_m$ is the coordinate transformation matrix from A_M to B_M. If there are no restrictions on the choice of A and B then $\mathbf{X}_m$ is a general nonsingular matrix, in which case the only solution to Eq. 1 is $\boldsymbol{\Delta} = \mathbf{0}$; but this is inconsistent with the definition of $\boldsymbol{\Delta}$. The conclusion we can draw is that it is impossible to preserve the invariance of any nonzero mapping without imposing some restrictions on the choice of bases. (This is a general result, not just a special property of motors.)

Given that the choice of bases must be restricted, let us now examine some specific sets of bases and the special forms that $\boldsymbol{\Delta}$ must take in order to be invariant within these sets.

Let us begin with a reciprocal basis pair comprising three unit rotations, three unit translations, three unit forces and three unit couples, all arranged along, about, or parallel to (as appropriate) the x, y and z axes of a right-handed Cartesian coordinate frame. The coordinate systems pertaining to this basis pair are known as Plücker ray and axis coordinates, and they are the two most commonly used coordinate systems for representing motors.

A basis pair that is defined in terms of a Cartesian coordinate frame, as above, is completely specified by the position and orientation of that frame. If we fix the position of the origin, but allow the frame to rotate freely about its origin, then we obtain a three-parameter set of reciprocal basis pairs with the property that the general form of $\mathbf{X}_m$ for any change of basis within the set is

$$\mathbf{X}_m = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix},$$

where $\mathbf{E}$ is a general 3×3 rotation matrix. With $\mathbf{X}_m$ restricted to this form, the most general solution to Eq. 1 is

$$\mathbf{\Delta} = \begin{bmatrix} \delta_1 \mathbf{1} & \delta_2 \mathbf{1} \\ \delta_3 \mathbf{1} & \delta_4 \mathbf{1} \end{bmatrix},$$

where δ_i are any four scalars satisfying $\delta_1\,\delta_4 \neq \delta_2\,\delta_3$, and $\mathbf{1}$ is a 3×3 identity matrix.

If we enlarge the set by allowing the frame to translate as well as rotate, then the general form of $\mathbf{X}_m$ is

$$\mathbf{X}_m = \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{E}\,\mathbf{S} & \mathbf{E} \end{bmatrix},$$

where $\mathbf{S}$ is a skew-symmetric matrix, and the general solution to Eq. 1 becomes

$$\mathbf{\Delta} = \delta_1 \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix} + \delta_2 \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}, \tag{2}$$

where $\delta_1 \neq 0$.

Now let us consider some transforms that do not work. One obvious example is uniform scaling, for which the only solution is $\mathbf{\Delta} = \mathbf{0}$. Most non-uniform scalings produce the same result. On the other hand, reflections and some special scalings do admit nonsingular solutions, but not necessarily in combination with other types of transform. For example, the general form of a reflection through the x–y, y–z or z–x plane for a Plücker-coordinate basis pair is

$$\mathbf{X}_m = \begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & -\mathbf{D} \end{bmatrix},$$

where $\mathbf{D}$ is a diagonal matrix with any permutation of $(1, -1, -1)$ on the main diagonal. If we want $\mathbf{\Delta}$ to be invariant with respect to reflections in addition to general translations and rotations of the Cartesian frame, then we must find out which values of δ_1 and δ_2 in Eq. 2 produce a matrix that is invariant to reflections. Unfortunately, the answer is that δ_1 must be zero; but this value is not allowed because it causes $\mathbf{\Delta}$ to be singular, and hence not a $1:1$ mapping. There is therefore no $1:1$ mapping between F^6 and M^6 that is simultaneously invariant to translations, rotations and reflections of the Cartesian coordinate frame.[1]

Another way to say this is that if you use the motor representation of motion and force vectors then you can not mix direct observations of a physical system with observations taken (knowingly) through a mirror. This limitation is an artifact of the representation.

[1] See also [4, footnote 7]. This is the observation that originally inspired this paper.

The above analysis covers only a small subset of the possible solutions to Eq. 1. It has the advantages of familiarity and relevance, since it deals with the most popular choice of coordinate systems for motors, but it is possible to synthesize many more solutions.

For example, consider the set $\Omega(A)$ of reciprocal basis pairs defined as follows: A is an arbitrary reciprocal basis pair, and $\Omega(A)$ is the set of all reciprocal basis pairs B with the property that the transform $\mathbf{X}_m$ from A_M to B_M happens to have the same numeric value as an orthonormal matrix. (The property of orthonormality is not defined in a dual system of vector spaces, but that does not prevent a transform matrix from being numerically equal to an orthonormal matrix.) $\Omega(A)$ is clearly a somewhat artificial construct, but it is nevertheless a 15-parameter set of bases with the property that any $\mathbf{\Delta}$ that is a scalar multiple of the identity matrix is invariant with respect to any change of basis within the set. Moreover, every reciprocal basis pair is a member of at least one of these sets.

There are almost certainly many more special solutions to Eq. 1 waiting to be described, but there is no need to pursue this line of investigation any further in the present context. One final comment that is worth making is that the invariant form of $\mathbf{\Delta}$ in Eq. 2 is entirely a function of the set of allowable basis pairs. Different sets have different invariant forms. So Eq. 2 is not as fundamental as it might appear to be.

The issue of invariance has received a lot of attention in the literature. The most relevant paper, at this point in the argument, is one by Lipkin and Duffy [8], which addresses the problem of how to construct kinestatic filters (for hybrid motion/force control) that are invariant with respect to specific changes of representation. Their paper uses Plücker axis coordinates for twists and Plücker ray coordinates for wrenches, both coordinate systems being constructed on a single underlying vector space. They have therefore chosen what amounts to a motor representation, as defined in this paper, together with the set of special basis pairs for Plücker coordinates. Eq. 2 is therefore the correct invariant form for their choice of bases, and it does indeed appear in their Eq. 39. (Incidentally, Plücker coordinates imply a specific choice for $\mathbf{\Delta}$, which is given by Eq. 2 with $\delta_1 = 1$ and $\delta_2 = 0$. This matrix appears as $\tilde{\Delta}$ in their paper.)

The important point is that Lipkin and Duffy (and, of course, many other papers) start with a given mathematical structure and a given set of allowable bases, and then proceed to identify invariant forms. The purpose of the present paper is to examine the initial choice of structures and basis sets by exploring the relationships between the choice of mathematical structure, the choice of intrinsic invariants, the choice of allowable basis sets, and the number of freedoms made available to the analyst.

4 Invariance in the Twist/Wrench Representation

This section investigates the restrictions that must be placed on the choice of bases when motion and force vectors are represented using a vector space of twists and a separate vector space of wrenches, these two spaces being connected only by a scalar product and a common-screw relation.

The purpose of this section is to demonstrate that the invariance problem described in the previous section is not confined to the motor representation, but is manifested in any representational scheme where there is a notion that both motion-type and force-type vectors arise from a common stock of type-neutral entities.

To make the point, the twist/wrench representation is defined to be a dual-space representation of twists and wrenches, to which has been added a common-screw relation between twists and wrenches. It is designed to be as close as possible to a pure dual-space representation without abandoning screw theory altogether. It is not in widespread use.

The twist/wrench representation consists of a dual system $\langle \mathsf{M}^6, \mathsf{F}^6, \cdot \rangle$ and a linear mapping $\mathbf{\Delta} : \mathsf{F}^6 \mapsto \mathsf{M}^6$. The elements of M^6 are twists, the elements of F^6 are wrenches, the scalar product is defined as the work performed by a wrench acting over a twist, and $\mathbf{\Delta}$ embodies the common-screw relation. Specifically, a twist $\mathbf{t}$ and a wrench $\mathbf{w}$ share a common screw axis if and only if

$$\mathbf{t} \propto \mathbf{\Delta}\, \mathbf{w} \,.$$

Notice that $\mathbf{t}$ shares a screw axis with all scalar multiples of $\mathbf{w}$, and vice versa. This is not a $1:1$ relation, so M^6 and F^6 can not be mapped together to form a single vector space. $\mathbf{\Delta}$ must be nonsingular, but it is also homogeneous: all scalar multiples of $\mathbf{\Delta}$ are equivalent.

On comparison with the motor representation, the only structural difference is that proportionality has replaced strict equality in the $\mathbf{\Delta}$ mapping. If we proceed to analyse the twist/wrench representation in the same way as the motor representation then we obtain the following equation in place of Eq. 1:

$$\mathbf{\Delta} \propto \mathbf{X}_m \, \mathbf{\Delta} \, \mathbf{X}_m^T \,.$$

As before, there is no solution to this equation in the general case, and it is therefore necessary to restrict the choice of bases. In fact, if the solutions to this equation are expressed in terms of an allowable set of transforms $\{\mathbf{X}_m\}$ and a set of mappings that are invariant within the transform set, then they are identical to the solutions to Eq. 1, except that the transform sets can be enlarged by including all scalar multiples of the elements already in the set.

The comments made earlier about the invariant forms of $\mathbf{\Delta}$ being a function of the choice of allowable bases, and about not being able to mix direct and mirror-image observations, apply also to the twist/wrench representation.

5 Consequences

The central theme of this paper is that motion and force vectors should be represented in separate vector spaces, and that the only relationship between these two spaces should be a duality relationship arising from a scalar product. If an additional relationship is introduced, like a common-screw or common-motor relation, then a conflict arises between, on the one hand, the need to preserve the invariance of both the scalar product and the common-entity relation with respect to a change of basis, and, on the other hand, the flexibility to provide a free choice of basis vectors. In this conflict, the preservation of invariance must take precedence.

In response to this situation, a theoretician might argue that one really ought to have complete freedom to choose a basis in a vector space, without violating the invariance of any relations or properties involving elements of that space, and hence argue that the motor and twist/wrench representations are imperfect mathematical models of the physical phenomena of motion and force.

On the other hand, a practitioner could accept this theoretical criticism, and counter with the observation that nobody uses any of the bases that don't work. A wily practitioner might also point out that you don't have a free choice of bases even in the dual-space representation since, having chosen a basis freely in one of the two spaces, the other basis is constrained to be reciprocal to it; so the argument is really about the degree of restriction, and whether it has any practical consequences.

Until recently, the only documented disadvantage of the motor representation compared with the dual-space representation has been the observation that a correct, invariant analysis of rigid-body freedoms and constraints (for the purpose of hybrid motion/force control of robots) is rather more complicated in the motor representation than in the dual-space representation [11].

There is now a second reason to prefer the dual-space representation. A new technique for analysing constrained rigid-body (and related) systems has recently been developed, and this technique has produced some new theoretical results and improvements in existing dynamics algorithms [6]. It is therefore arguably of some practical value. The technique relies on being able to partition both M^n and F^n simultaneously, each into two subspaces, these being aligned with the freedoms and constraints of the particular system under investigation. In the dual-space representation, a new basis pair is constructed that is simultaneously in alignment with both partitions. The analysis then proceeds in the new basis pair, where the constraints are trivial. This technique can not be used in the motor or twist/wrench representations because there are not enough freedoms available to choose a basis pair with the desired special properties.

The argument at work here is analogous to the argument against the 4×4 matrix representation used by some authors to derive and express the equations of motion of a rigid-body system (*e.g.*, [7]). This representation is perfectly adequate for the job, but fails to support certain useful tools. For example, an inertia matrix in this representation contains ten independent parameters, which is enough to represent a general rigid-body inertia, but not enough to represent a general articulated-body inertia. So anyone who uses the 4×4 matrix representation must do without articulated-body inertias and the concepts, techniques and algorithms that they support.

In the field of hybrid motion/force control, there has been much interest in invariant techniques for describing and analysing motion and force freedoms and constraints. One idea that is currently popular is that if the concepts used during the analysis have physical meaning then the final result will be invariant [3]. This paper illustrates a potential weakness in this strategy: by augmenting the dual-space representation with the common-screw relation, which is certainly a physically-meaningful concept, the result is a reduction in the anaylst's freedom to write invariant formulas.

6 Conclusion

This paper has presented an analysis of an invariance problem involving the following quantities:

1. a space of motion vectors representing infinitesimal displacements, velocities, etc. of a physical system in general, and a rigid body in particular,

2. a space of force vectors representing the forces acting on the same system,

3. a scalar product defined between motion and force vectors, and

4. an additional relationship defined between motion and force vectors.

The scalar product represents the work (power, etc.) done by a force acting over a displacement (velocity, etc.), and is expected to be invariant with respect to the choice of basis vectors. The additional relationship represents a common property that is a consequence of the method of defining certain kinds of motion and force vector, and is also expected to be invariant.

Three possible representations of motion and force vectors are considered: dual-space, motor and twist/wrench. In the dual-space representation, motion vectors are elements of a vector space M^n, force vectors are elements of a separate vector space F^n, and the two spaces, together with the scalar product defined between them, constitute a dual system of vector spaces. In this representation, the scalar product is invariant in value and form if the two sets of basis vectors (one basis in each space) form a reciprocal basis pair. This condition allows one basis to be chosen arbitrarily, whereupon the other basis is uniquely defined by the reciprocity conditions.

The dual-space representation does not define any additional relationship between motions and forces. In contrast, the motor representation defines motion and force vectors to be motors. This creates a $1:1$ common-motor relationship between them. The twist/wrench representation is essentially the same as the dual-space representation, except that a common-screw relationship is defined between twists (motion vectors) and wrenches (force vectors) with a common screw axis.

This paper shows that if a representation includes an additional relationship then additional restrictions must be placed on the choice of basis vectors in order to preserve the invariance of the additional relationship. These restrictions can be inconvenient, and they reduce the set of analytical tools that can be supported by the representation. It is argued that this is a good reason to prefer the dual-space representation.

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