Plücker Basis Vectors

Roy Featherstone
Department of Information Engineering
The Australian National University
Canberra, ACT 0200, Australia
Email: Roy.Featherstone@anu.edu.au

***Abstract*— 6-D vectors are routinely expressed in Plücker coordinates; yet there is almost no mention in the literature of the basis vectors that give rise to these coordinates. This paper identifies the Plücker basis vectors, and uses them to explain the following: the relationship between a 6-D vector and its Plücker coordinates, the relationship between a 6-D vector and the pair of 3-D vectors used to define it, and the correct way to differentiate a 6-D vector in a moving coordinate system.**

I. Introduction

6-D vectors are used to describe the motions of rigid bodies and the forces acting upon them. They are therefore useful for describing the kinematics and dynamics of rigid-body systems in general, and robot mechanisms in particular. 6-D vectors come in various forms, such as twists, wrenches, motors, spatial vectors, Lie-algebra elements, and simple concatenations of pairs of 3-D vectors. For examples, see [1], [2], [3], [4], [6], [8], [9], [10], [12], [16]. Nearly all such vectors are expressed using Plücker coordinates—a system of coordinates invented in the 1860s by J. Plücker [11].

It is a basic tenet of linear algebra that a coordinate system on a vector space is defined by a basis. It therefore follows that Plücker coordinates must be defined by a basis. Yet, despite the widespread use of Plücker coordinates, there appears to be almost no mention of the basis vectors that give rise to them. The only example the author could find is the standard basis described in [4].

The role of a basis is to define the relationship between the coordinates and the vectors they represent. Without an explicit definition of the Plücker basis vectors, there is not a clear description of the relationship between Plücker coordinates and the 6-D vectors they represent. This has occasionally led to confusion over the true nature of 6-D vectors, as evidenced by the recent debate on the definition of the 6-D acceleration vector [5], [7], [13], [14], [15]. There is a tendency to regard 6-D vectors as being ordered pairs of 3-D vectors, but this model is inaccurate, as it does not properly take into account the role of the reference point.

This paper makes the following contributions: it defines the Plücker basis vectors; it explains the relationship between a Plücker coordinate vector and the 6-D vector it represents; it explains the relationship between a 6-D vector and the two 3-D vectors that define it; and it shows how to differentiate a 6-D vector in a moving Plücker coordinate system, using acceleration as an example. For the sake of a concrete exposition, this paper uses the notation and terminology of spatial vectors; but the results apply generally to any 6-D vector that is expressed using Plücker coordinates.

The rest of this paper is organized as follows. First, the Plücker basis vectors are described. Then the topic of dual coordinate systems is discussed. This is relevant to those 6-D vector formalisms in which force vectors are deemed to occupy a different vector space to motion vectors. Next, a convenient operator notation is introduced, that allows us to express and manipulate the mappings between coordinate vectors and the vectors they represent, as determined by the basis vectors. The paper then proceeds to examine the nature of the relationship between spatial vectors and the pairs of Euclidean vectors that are used to define them. Finally, the method of differentiation in a moving Plücker coordinate system is explained, and the results used to illuminate the relationship between competing definitions of 6-D acceleration vectors.

II. Plücker Basis Vectors

Different kinds of vector belong to different vector spaces. We therefore begin by defining the vector spaces E^n and R^n for n-dimensional Euclidean and coordinate vectors, respectively. Elements of E^n have properties of magnitude and direction, while elements of R^n are n-tuples of real numbers.

Spatial vectors are not Euclidean, and therefore do not belong in E^6. Furthermore, it is useful to maintain a distinction between those vectors that describe the motions of rigid bodies and those that describe the forces acting upon them. We therefore define two vector spaces, M^6 and F^6, one for spatial motion vectors and one for spatial forces. Elements of M^6 describe velocities, accelerations, directions of motion freedom, and so on. Elements of F^6 describe forces, momenta, impulses, and so on.

Spatial vectors are usually constructed from pairs of 3D Euclidean vectors. Let us now examine how this is done. In particular, let us examine the construction of a velocity vector and a force vector.

Referring to Figure 1, the velocity of a rigid body can be specified by a pair of vectors, $\boldsymbol{\omega}, \boldsymbol{v}_O \in \mathsf{E}^3$, where $\boldsymbol{\omega}$ is the angular velocity of the body as a whole, and $\boldsymbol{v}_O$ is the linear

This paper appeared in Proc. IEEE Int. Conf. Robotics & Automation, Orlando, FL, May 15–19 2006, pp. 1892–1897.

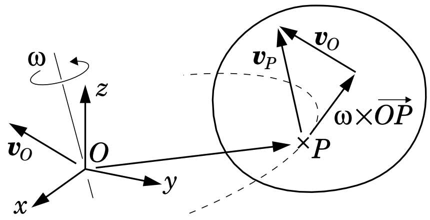

Fig. 1. Rigid body velocity

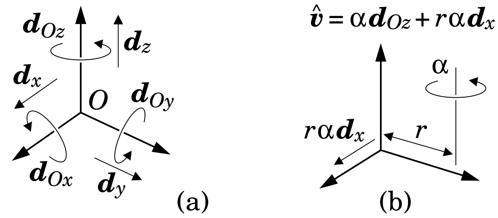

Fig. 2. Plücker motion basis (a), and example (b)

velocity of the body-fixed point that is passing through an arbitrary given point, O, in the underlying physical space. The rigid-body velocity described by these two vectors is the sum of a pure rotation, in which the body rotates with angular velocity $\boldsymbol{\omega}$ about an axis passing through O, and a pure translation given by $\boldsymbol{v}_O$. This can be seen in the formula

$$\boldsymbol{v}_P = \boldsymbol{v}_O + \boldsymbol{\omega} \times \overrightarrow{OP}\,, \tag{1}$$

which gives the velocity of the body-fixed point at P. The right-hand side is the sum of a component due to the translation of the whole rigid body by $\boldsymbol{v}_O$, and a component due to its rotation by $\boldsymbol{\omega}$ about an axis passing through O.

Observe how the meanings change as the two vectors are combined: $\boldsymbol{\omega}$ on its own is a disembodied angular velocity that is the same for any choice of O, and $\boldsymbol{v}_O$ on its own refers specifically to the one point in the body that coincides with O at the current instant; but when we combine the two, the rigid-body velocity they describe is the sum of a rotation specifically about an axis passing through O, and a pure translation in which every point in the body travels with velocity $\boldsymbol{v}_O$.

Let us introduce a Cartesian coordinate frame, $Oxyz$, with its origin at O. This frame defines three mutually perpendicular directions, x, y and z. These directions allow us to define an orthonormal basis,

$$\mathcal{C} = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \subset \mathsf{E}^3\,, \tag{2}$$

in which the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ point in the x, y and z directions, respectively. This basis gives rise to a Cartesian coordinate system on E^3, such that $\boldsymbol{\omega}$ and $\boldsymbol{v}_O$ can be expressed in terms of their Cartesian coordinates:

$$\boldsymbol{\omega} = \omega_x\,\boldsymbol{i} + \omega_y\,\boldsymbol{j} + \omega_z\,\boldsymbol{k} \tag{3}$$

and

$$\boldsymbol{v}_O = v_{Ox}\,\boldsymbol{i} + v_{Oy}\,\boldsymbol{j} + v_{Oz}\,\boldsymbol{k}\,. \tag{4}$$

We can now say that the Euclidean vectors $\boldsymbol{\omega}$ and $\boldsymbol{v}_O$ are represented by the coordinate vectors[1] $\underline{\boldsymbol{\omega}} = [\,\omega_x\ \omega_y\ \omega_z\,]^T \in \mathsf{R}^3$ and $\underline{\boldsymbol{v}}_O = [\,v_{Ox}\ v_{Oy}\ v_{Oz}\,]^T \in \mathsf{R}^3$, respectively, in the coordinate system defined by the basis $\mathcal{C}$.

It is well known that the six numbers $\omega_x, \ldots, v_{Oz}$ are the Plücker coordinates of a spatial vector,[2] $\hat{\boldsymbol{v}} \in \mathsf{M}^6$, that represents the same rigid-body velocity as the two 3D vectors $\boldsymbol{\omega}$ and $\boldsymbol{v}_O$. To establish the relationship between $\hat{\boldsymbol{v}}$ and its coordinates, we define the following basis on M^6:

$$\mathcal{D}_O = \{\boldsymbol{d}_{Ox}, \boldsymbol{d}_{Oy}, \boldsymbol{d}_{Oz}, \boldsymbol{d}_x, \boldsymbol{d}_y, \boldsymbol{d}_z\} \subset \mathsf{M}^6\,, \tag{5}$$

in which $\boldsymbol{d}_{Ox}$, $\boldsymbol{d}_{Oy}$ and $\boldsymbol{d}_{Oz}$ are unit rotations about the directed lines Ox, Oy and Oz (which pass through O in the x, y and z directions, respectively), and $\boldsymbol{d}_x$, $\boldsymbol{d}_y$ and $\boldsymbol{d}_z$ are unit translations in the x, y and z directions (see Figure 2(a)). Thus, if the body were rotating about Ox with an angular velocity of magnitude α, then its spatial velocity would be $\alpha\,\boldsymbol{d}_{Ox}$. Likewise, if the body were translating with a linear velocity of $\boldsymbol{v}_O$, then its spatial velocity would be $v_{Ox}\,\boldsymbol{d}_x + v_{Oy}\,\boldsymbol{d}_y + v_{Oz}\,\boldsymbol{d}_z$. Note the difference between this expression and the one in (4): the former is a spatial vector (i.e., an element of M^6), and the latter a Euclidean vector (an element of E^3). A third example is shown in Figure 2(b). This example shows a rotation of magnitude α about an axis that is parallel to the z axis and passes through the point $(0, r, 0)$. This motion is represented by the spatial vector $\alpha\boldsymbol{d}_{Oz} + r\alpha\boldsymbol{d}_x$. Observe that the translational component equals the velocity of a particle at the origin that is rotating about $(0, r, 0)$ with an angular velocity of α.

It can be seen, by inspection, that the spatial vector

$$\hat{\boldsymbol{v}} = \omega_x\,\boldsymbol{d}_{Ox} + \omega_y\,\boldsymbol{d}_{Oy} + \omega_z\,\boldsymbol{d}_{Oz} + v_{Ox}\,\boldsymbol{d}_x + v_{Oy}\,\boldsymbol{d}_y + v_{Oz}\,\boldsymbol{d}_z \tag{6}$$

represents the same rigid-body velocity as that described by the two vectors $\boldsymbol{\omega}$ and $\boldsymbol{v}_O$ above. We may therefore conclude that $\mathcal{D}_O$ is the basis that gives rise to the Plücker coordinate system in M^6 associated with $Oxyz$, and that the coordinate vector

$$\underline{\hat{\boldsymbol{v}}}_O = [\,\omega_x\ \omega_y\ \omega_z\ v_{Ox}\ v_{Oy}\ v_{Oz}\,]^T \in \mathsf{R}^6 \tag{7}$$

represents the spatial vector $\hat{\boldsymbol{v}} \in \mathsf{M}^6$ in the Plücker coordinate system defined by the basis $\mathcal{D}_O$. Equation (7) is often written in the form

$$\underline{\hat{\boldsymbol{v}}}_O = \begin{bmatrix} \underline{\boldsymbol{\omega}} \\ \underline{\boldsymbol{v}}_O \end{bmatrix}, \tag{8}$$

in which the right-hand side is the concatenation of the two coordinate vectors $\underline{\boldsymbol{\omega}}$ and $\underline{\boldsymbol{v}}_O$.

Observe the pattern of subscripts in (6). Each quantity that depends on the location of O contains an O in its subscript. Note, however, that the subscript in $\boldsymbol{d}_{Ox}$ refers to the line Ox, whereas the subscript in v_{Ox} is really two subscripts run together, since v_{Ox} is the x coordinate of $\boldsymbol{v}_O$. Although

[1] Coordinate vectors are underlined to distinguish them from the vectors they represent.

[2] Spatial vectors other than basis vectors are marked with a hat. Basis vectors are left unmarked.

individual terms may vary, it can be shown that the complete expression on the right-hand side of (6) is invariant with respect to both the position and orientation of $Oxyz$. Thus, $\hat{\boldsymbol{v}}$ is a genuinely invariant representation of rigid-body velocity.

A spatial force vector is constructed in a similar manner. Any system of applied forces acting on a single rigid body is equivalent to a single resultant force vector, $\boldsymbol{f}$, together with a moment vector, $\boldsymbol{n}_O$, giving the moment of the force system about an arbitrary given point, O. Although $\boldsymbol{f}$ itself is independent of O, the quantity it represents is a force acting on the rigid body along a line passing through O.

Introducing the coordinate frame $Oxyz$, and the basis $\mathcal{C}$, we can express $\boldsymbol{f}$ and $\boldsymbol{n}_O$ in terms of their Cartesian coordinates:

$$\boldsymbol{f} = f_x\,\boldsymbol{i} + f_y\,\boldsymbol{j} + f_z\,\boldsymbol{k} \tag{9}$$

and

$$\boldsymbol{n}_O = n_{Ox}\,\boldsymbol{i} + n_{Oy}\,\boldsymbol{j} + n_{Oz}\,\boldsymbol{k}\,. \tag{10}$$

As before, the six numbers $n_{Ox}, \ldots, f_z$ are the Plücker coordinates of a spatial force vector, $\hat{\boldsymbol{f}} \in \mathsf{F}^6$, representing the same force system as $\boldsymbol{f}$ and $\boldsymbol{n}_O$. To establish the relationship between $\hat{\boldsymbol{f}}$ and its coordinates, we define the following basis on F^6:

$$\mathcal{E}_O = \{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z, \boldsymbol{e}_{Ox}, \boldsymbol{e}_{Oy}, \boldsymbol{e}_{Oz}\} \subset \mathsf{F}^6\,, \tag{11}$$

in which $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$ are unit couples in the x, y and z directions, and $\boldsymbol{e}_{Ox}$, $\boldsymbol{e}_{Oy}$ and $\boldsymbol{e}_{Oz}$ are unit forces along the lines Ox, Oy and Oz. Again, it can be seen, by inspection, that the spatial vector

$$\hat{\boldsymbol{f}} = n_{Ox}\,\boldsymbol{e}_x + n_{Oy}\,\boldsymbol{e}_y + n_{Oz}\,\boldsymbol{e}_z + f_x\,\boldsymbol{e}_{Ox} + f_y\,\boldsymbol{e}_{Oy} + f_z\,\boldsymbol{e}_{Oz} \tag{12}$$

represents the same force system as the two vectors $\boldsymbol{f}$ and $\boldsymbol{n}_O$. We may therefore conclude that $\mathcal{E}_O$ is the basis that gives rise to the Plücker coordinate system in F^6 associated with $Oxyz$, and that the coordinate vector

$$\underline{\hat{\boldsymbol{f}}}_O = [\,n_{Ox}\;n_{Oy}\;n_{Oz}\;f_x\;f_y\;f_z\,]^T \in \mathsf{R}^6 \tag{13}$$

represents the spatial vector $\hat{\boldsymbol{f}} \in \mathsf{F}^6$ in the Plücker coordinate system defined by the basis $\mathcal{E}_O$. Equation (13) can be written in the form

$$\underline{\hat{\boldsymbol{f}}}_O = \begin{bmatrix} \underline{\boldsymbol{n}}_O \\ \underline{\boldsymbol{f}} \end{bmatrix}, \tag{14}$$

in which the right-hand side is the concatenation of the two coordinate vectors $\underline{\boldsymbol{n}}_O$ and $\underline{\boldsymbol{f}}$.

III. Dual Coordinate Systems

Neither M^6 nor F^6 defines an inner product on its elements. Instead, there is a scalar product that takes one argument from each space and produces a real number representing work. Thus, if $\hat{\boldsymbol{m}} \in \mathsf{M}^6$ and $\hat{\boldsymbol{f}} \in \mathsf{F}^6$, then $\hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{m}}$ is the work done by a force $\hat{\boldsymbol{f}}$ acting on a rigid body moving with motion $\hat{\boldsymbol{m}}$. For convenience, we define $\hat{\boldsymbol{m}} \cdot \hat{\boldsymbol{f}}$ to mean the same as $\hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{m}}$, but the expressions $\hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{f}}$ and $\hat{\boldsymbol{m}} \cdot \hat{\boldsymbol{m}}$ are not defined.

Let $\mathcal{D} = \{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_6\}$ be an arbitrary basis on M^6. For any choice of $\mathcal{D}$, there exists a unique basis $\mathcal{E} = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_6\}$ on F^6 with the property that

$$\boldsymbol{d}_i \cdot \boldsymbol{e}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

$\mathcal{E}$ is the dual (or reciprocal) basis to $\mathcal{D}$, and vice versa. The pair $(\mathcal{D}, \mathcal{E})$ can be called a dual basis pair, or simply a dual basis. It defines a dual coordinate system encompassing both M^6 and F^6, in which elements of M^6 are expressed via $\mathcal{D}$ and elements of F^6 via $\mathcal{E}$. A dual coordinate system is the spatial-vector equivalent of a Cartesian coordinate system in a Euclidean vector space.

In any dual coordinate system, the following equations hold:

$$\hat{\boldsymbol{m}} \cdot \hat{\boldsymbol{f}} = \underline{\hat{\boldsymbol{m}}}^T\,\underline{\hat{\boldsymbol{f}}}\,, \tag{16}$$

$$\boldsymbol{e}_i \cdot \hat{\boldsymbol{m}} = m_i \tag{17}$$

and

$$\boldsymbol{d}_i \cdot \hat{\boldsymbol{f}} = f_i\,, \tag{18}$$

where $\underline{\hat{\boldsymbol{m}}}$ and $\underline{\hat{\boldsymbol{f}}}$ are the coordinate vectors representing $\hat{\boldsymbol{m}} \in \mathsf{M}^6$ and $\hat{\boldsymbol{f}} \in \mathsf{F}^6$ in bases $\mathcal{D}$ and $\mathcal{E}$, respectively, and m_i and f_i are the individual coordinates. These results follow directly from (15). If A and B are any two dual coordinate systems, and ${}^B\boldsymbol{X}_A^{\mathsf{M}}$ is the coordinate transformation matrix from A to B coordinates for motion vectors, then the corresponding transformation matrix for force vectors is

$${}^B\boldsymbol{X}_A^{\mathsf{F}} = ({}^B\boldsymbol{X}_A^{\mathsf{M}})^{-T}\,. \tag{19}$$

This equation follows from the invariance property of the scalar product, which can be expressed as

$$\underline{\hat{\boldsymbol{m}}}_A^T\,\underline{\hat{\boldsymbol{f}}}_A = \underline{\hat{\boldsymbol{m}}}_B^T\,\underline{\hat{\boldsymbol{f}}}_B \tag{20}$$

for all $\hat{\boldsymbol{m}}$, $\hat{\boldsymbol{f}}$, A and B.

The Plücker bases in (5) and (11) satisfy (15), so the basis pair $(\mathcal{D}_O, \mathcal{E}_O)$ defines a dual coordinate system on M^6 and F^6.

It is possible to write the elements of $\mathcal{D}_O$ in a different order, provided one does the same to $\mathcal{E}_O$. The result is a reordering of the cordinates in the coordinate vectors. For example, we could rewrite $\mathcal{D}_O$ and $\mathcal{E}_O$ as

$$\mathcal{D}_O = \{\boldsymbol{d}_x, \boldsymbol{d}_y, \boldsymbol{d}_z, \boldsymbol{d}_{Ox}, \boldsymbol{d}_{Oy}, \boldsymbol{d}_{Oz}\}$$

and

$$\mathcal{E}_O = \{\boldsymbol{e}_{Ox}, \boldsymbol{e}_{Oy}, \boldsymbol{e}_{Oz}, \boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}\,,$$

in which case the coordinate vectors representing $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{f}}$ would be

$$\underline{\hat{\boldsymbol{v}}}_O = [\,v_{Ox}\;v_{Oy}\;v_{Oz}\;\omega_x\;\omega_y\;\omega_z\,]^T$$

and

$$\underline{\hat{\boldsymbol{f}}}_O = [\,f_x\;f_y\;f_z\;n_{Ox}\;n_{Oy}\;n_{Oz}\,]^T\,.$$

Some authors prefer this linear-before-angular ordering to the angular-before-linear ordering in (7) and (13). With the aid of Plücker bases, it is immediately obvious that the difference between these two orderings is purely cosmetic—they both represent the same spatial vectors.

IV. Basis Mappings

Suppose $\mathcal{B} = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ is a basis on a vector space U. Given $\mathcal{B}$, we can express any vector $\boldsymbol{u} \in U$ in the form

$$\boldsymbol{u} = \sum_{i=1}^{n} \boldsymbol{b}_i \, u_i \,,$$

where u_i are the coordinates of $\boldsymbol{u}$ in $\mathcal{B}$. This equation can also be written in the form

$$\boldsymbol{u} = \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \boldsymbol{\mathcal{B}} \, \underline{\boldsymbol{u}} \,, \tag{21}$$

where $\boldsymbol{\mathcal{B}}$ is the operator that maps coordinate vectors to the vectors they represent in the basis $\mathcal{B}$. We therefore call $\boldsymbol{\mathcal{B}}$ the basis mapping associated with $\mathcal{B}$. Formally, $\boldsymbol{\mathcal{B}}$ is a mapping from R^n to U defined as follows:

$$\boldsymbol{\mathcal{B}} : \mathsf{R}^n \mapsto U : \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \mapsto \sum_{i=1}^{n} \boldsymbol{b}_i \, u_i \,. \tag{22}$$

If $\boldsymbol{\mathcal{B}}$ is written as a $1 \times n$ array of basis vectors, as shown in (21), then the action of $\boldsymbol{\mathcal{B}}$ on $\underline{\boldsymbol{u}}$ can be understood as the result of a formal matrix multiplication between the two.

$\boldsymbol{\mathcal{B}}$ is a $1:1$ mapping, and is therefore invertible; so there must exist an inverse mapping, $\boldsymbol{\mathcal{B}}^{-1}$, that satisfies $\underline{\boldsymbol{u}} = \boldsymbol{\mathcal{B}}^{-1} \boldsymbol{u}$. A formal expression for $\boldsymbol{\mathcal{B}}^{-1}$ can be stated as

$$\boldsymbol{\mathcal{B}}^{-1} = \begin{bmatrix} \boldsymbol{b}_1^* \cdot \\ \vdots \\ \boldsymbol{b}_n^* \cdot \end{bmatrix} , \tag{23}$$

where $\{\boldsymbol{b}_1^*, \dots, \boldsymbol{b}_n^*\}$ is the dual basis to $\mathcal{B}$, and $\boldsymbol{b}_i^* \cdot$ is the operator that maps any vector $\boldsymbol{u} \in U$ to the scalar $\boldsymbol{b}_i^* \cdot \boldsymbol{u}$. Expanding $\boldsymbol{\mathcal{B}}^{-1} \boldsymbol{\mathcal{B}}$ gives

$$\boldsymbol{\mathcal{B}}^{-1} \boldsymbol{\mathcal{B}} = \begin{bmatrix} \boldsymbol{b}_1^* \cdot \\ \vdots \\ \boldsymbol{b}_n^* \cdot \end{bmatrix} \begin{bmatrix} \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{b}_1^* \cdot \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_1^* \cdot \boldsymbol{b}_n \\ \vdots & \ddots & \vdots \\ \boldsymbol{b}_n^* \cdot \boldsymbol{b}_1 & \cdots & \boldsymbol{b}_n^* \cdot \boldsymbol{b}_n \end{bmatrix}$$

which equates to the identity matrix because of the reciprocity condition (15). Special cases of interest are:

$$\boldsymbol{\mathcal{C}}^{-1} = \begin{bmatrix} \boldsymbol{i} \cdot \\ \boldsymbol{j} \cdot \\ \boldsymbol{k} \cdot \end{bmatrix} , \tag{24}$$

$$\boldsymbol{\mathcal{D}}_O^{-1} = \begin{bmatrix} \boldsymbol{e}_x \cdot \\ \vdots \\ \boldsymbol{e}_{Oz} \cdot \end{bmatrix} \tag{25}$$

and

$$\boldsymbol{\mathcal{E}}_O^{-1} = \begin{bmatrix} \boldsymbol{d}_{Ox} \cdot \\ \vdots \\ \boldsymbol{d}_z \cdot \end{bmatrix} . \tag{26}$$

Basis mappings provide a simple but powerful tool for expressing the relationships between vectors. To illustrate their use, consider the task of formulating the transformation matrix between two coordinate systems, A and B. Let $\underline{\boldsymbol{u}}_A$ and $\underline{\boldsymbol{u}}_B$ be the coordinate vectors representing the vector $\boldsymbol{u} \in U$ in A and B coordinates. If ${}^B\boldsymbol{X}_A$ is the coordinate transformation matrix from A to B, then we have

$$\underline{\boldsymbol{u}}_B = {}^B\boldsymbol{X}_A \, \underline{\boldsymbol{u}}_A \,.$$

However, if $\boldsymbol{\mathcal{B}}_A$ and $\boldsymbol{\mathcal{B}}_B$ are the basis maps for A and B, then we also have

$$\underline{\boldsymbol{u}}_B = \boldsymbol{\mathcal{B}}_B^{-1} \boldsymbol{u} = \boldsymbol{\mathcal{B}}_B^{-1} \, \boldsymbol{\mathcal{B}}_A \, \underline{\boldsymbol{u}}_A \,,$$

so

$${}^B\boldsymbol{X}_A = \boldsymbol{\mathcal{B}}_B^{-1} \, \boldsymbol{\mathcal{B}}_A \,.$$

Expanding this equation gives

$$\begin{aligned} {}^B\boldsymbol{X}_A &= \boldsymbol{\mathcal{B}}_B^{-1} \begin{bmatrix} \boldsymbol{b}_{A1} & \cdots & \boldsymbol{b}_{An} \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{\mathcal{B}}_B^{-1} \boldsymbol{b}_{A1} & \cdots & \boldsymbol{\mathcal{B}}_B^{-1} \boldsymbol{b}_{An} \end{bmatrix} ; \end{aligned}$$

but $\boldsymbol{\mathcal{B}}_B^{-1} \, \boldsymbol{b}_{Ai}$ is just the coordinate vector representing $\boldsymbol{b}_{Ai}$ in B coordinates, so we may conclude that ${}^B\boldsymbol{X}_A$ is a square matrix whose columns are the coordinates of the old basis vectors in the new coordinate system. (This is a standard result. The point is simply the speed with which it can be obtained.)

V. Relationship Between Spatial and Euclidean Vectors

Let us examine the relationship between a spatial vector and the pair of Euclidean vectors that are used to define it. If we partition $\mathcal{D}_O$ into two sub-bases, $\mathcal{D}_O^{rot} = \{\boldsymbol{d}_{Ox}, \boldsymbol{d}_{Oy}, \boldsymbol{d}_{Oz}\}$ and $\mathcal{D}_O^{lin} = \{\boldsymbol{d}_x, \boldsymbol{d}_y, \boldsymbol{d}_z\}$, then (6) can be written as follows:

$$\begin{aligned} \hat{\boldsymbol{v}} &= \boldsymbol{\mathcal{D}}_O^{rot} \, \underline{\boldsymbol{\omega}} + \boldsymbol{\mathcal{D}}_O^{lin} \, \underline{\boldsymbol{v}}_O \\ &= \boldsymbol{\mathcal{D}}_O^{rot} \, \boldsymbol{\mathcal{C}}^{-1} \, \boldsymbol{\omega} + \boldsymbol{\mathcal{D}}_O^{lin} \, \boldsymbol{\mathcal{C}}^{-1} \, \boldsymbol{v}_O \\ &= \boldsymbol{Rot}_O^{\mathsf{M}} \, \boldsymbol{\omega} + \boldsymbol{Lin}^{\mathsf{M}} \, \boldsymbol{v}_O \,, \end{aligned} \tag{27}$$

where

$$\boldsymbol{Rot}_O^{\mathsf{M}} = \boldsymbol{\mathcal{D}}_O^{rot} \, \boldsymbol{\mathcal{C}}^{-1} = \boldsymbol{d}_{Ox} \, \boldsymbol{i} \cdot + \boldsymbol{d}_{Oy} \, \boldsymbol{j} \cdot + \boldsymbol{d}_{Oz} \, \boldsymbol{k} \cdot \tag{28}$$

and

$$\boldsymbol{Lin}^{\mathsf{M}} = \boldsymbol{\mathcal{D}}_O^{lin} \, \boldsymbol{\mathcal{C}}^{-1} = \boldsymbol{d}_x \, \boldsymbol{i} \cdot + \boldsymbol{d}_y \, \boldsymbol{j} \cdot + \boldsymbol{d}_z \, \boldsymbol{k} \cdot \,. \tag{29}$$

Expressions like $\boldsymbol{d}_{Ox} \, \boldsymbol{i} \cdot$ are dyads that map Euclidean vectors to spatial motion vectors. $\boldsymbol{d}_{Ox} \, \boldsymbol{i} \cdot$ maps any vector $\boldsymbol{v} \in \mathsf{E}^3$ to $\boldsymbol{d}_{Ox} \, (\boldsymbol{i} \cdot \boldsymbol{v}) = \boldsymbol{d}_{Ox} \, v_x \in \mathsf{M}^6$, and so on. The operators $\boldsymbol{Rot}_O^{\mathsf{M}}$ and $\boldsymbol{Lin}^{\mathsf{M}}$ are therefore both dyadics (sums of dyads). $\boldsymbol{Rot}_O^{\mathsf{M}}$ maps Euclidean vectors to the set of pure rotations about axes passing through O; and $\boldsymbol{Lin}^{\mathsf{M}}$ maps Euclidean vectors to the set of pure translations. It can be shown that both $\boldsymbol{Rot}_O^{\mathsf{M}}$ and $\boldsymbol{Lin}^{\mathsf{M}}$ are independent of the orientation of the coordinate frame, $Oxyz$, that gave rise to the bases $\mathcal{C}$ and $\mathcal{D}_O$. Furthermore, $\boldsymbol{Lin}^{\mathsf{M}}$ is also independent of the position of O. Therefore, $\boldsymbol{Lin}^{\mathsf{M}}$ is an invariant tensor, while $\boldsymbol{Rot}_O^{\mathsf{M}}$ depends only on the position of O.

A similar analysis can be performed for force vectors:

$$\begin{aligned} \hat{\boldsymbol{f}} &= \boldsymbol{\mathcal{E}}_O^{rot} \, \underline{\boldsymbol{n}}_O + \boldsymbol{\mathcal{E}}_O^{lin} \, \underline{\boldsymbol{f}} \\ &= \boldsymbol{\mathcal{E}}_O^{rot} \, \boldsymbol{\mathcal{C}}^{-1} \, \boldsymbol{n}_O + \boldsymbol{\mathcal{E}}_O^{lin} \, \boldsymbol{\mathcal{C}}^{-1} \, \boldsymbol{f} \\ &= \boldsymbol{Rot}^{\mathsf{F}} \, \boldsymbol{n}_O + \boldsymbol{Lin}_O^{\mathsf{F}} \, \boldsymbol{f} \,, \end{aligned} \tag{30}$$

where

$$\boldsymbol{Rot}^{\mathsf{F}} = \boldsymbol{e}_x\,\boldsymbol{i}\cdot + \boldsymbol{e}_y\,\boldsymbol{j}\cdot + \boldsymbol{e}_z\,\boldsymbol{k}\cdot \tag{31}$$

and

$$\boldsymbol{Lin}_O^{\mathsf{F}} = \boldsymbol{e}_{Ox}\,\boldsymbol{i}\cdot + \boldsymbol{e}_{Oy}\,\boldsymbol{j}\cdot + \boldsymbol{e}_{Oz}\,\boldsymbol{k}\cdot\,. \tag{32}$$

It is not accurate to describe $\boldsymbol{\omega}$, $\boldsymbol{v}_O$, $\boldsymbol{n}_O$ and $\boldsymbol{f}$ as being the angular and linear components of $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{f}}$. However, it is possible to regard them as the vector-valued coordinates of $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{f}}$ in the coordinate systems defined by the basis tensors $\boldsymbol{Rot}_O^{\mathsf{M}}$, $\boldsymbol{Lin}^{\mathsf{M}}$, $\boldsymbol{Rot}^{\mathsf{F}}$ and $\boldsymbol{Lin}_O^{\mathsf{F}}$.

In summary, the mapping from $\mathsf{E}^3 \times \mathsf{E}^3$ to either M^6 or F^6 is accomplished by a pair of dyadic tensors, one of which is invariant, while the other is a function of the location of the reference point, O, that was used when specifying the two Euclidean vectors.

VI. Differentiation

Let $\boldsymbol{u}(t)$ be a vector-valued, differentiable function of a real variable t. The derivative of $\boldsymbol{u}$ with respect to t is itself a vector, and is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\boldsymbol{u}(t) = \lim_{\delta t\to 0} \frac{\boldsymbol{u}(t+\delta t) - \boldsymbol{u}(t)}{\delta t}\,. \tag{33}$$

This equation is valid for any variable t; but we will assume below that t denotes time, and we will use the standard dot notation for time derivatives ($\mathrm{d}\boldsymbol{u}/\mathrm{d}t = \dot{\boldsymbol{u}}$, etc.).

Equation (33) applies to all vectors, including coordinate vectors. It therefore follows that the derivative of a coordinate vector is its component-wise derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix} = \lim_{\delta t\to 0}\frac{1}{\delta t}\begin{bmatrix} u_1(t+\delta t) - u_1(t) \\ \vdots \\ u_n(t+\delta t) - u_n(t) \end{bmatrix} = \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix}. \tag{34}$$

Let $\mathcal{B}$ be a basis on U, and let $\underline{\boldsymbol{u}}$ be the coordinate vector that represents the vector $\boldsymbol{u} \in U$ in $\mathcal{B}$ coordinates. The derivatives of $\underline{\boldsymbol{u}}$ and $\boldsymbol{u}$ are $\dot{\underline{\boldsymbol{u}}}$ and $\dot{\boldsymbol{u}}$, respectively, but the coordinate vector that represents $\dot{\boldsymbol{u}}$ in $\mathcal{B}$ is $\boldsymbol{\mathcal{B}}^{-1}\,\dot{\boldsymbol{u}}$. The relationship between $\dot{\underline{\boldsymbol{u}}}$ and $\boldsymbol{\mathcal{B}}^{-1}\,\dot{\boldsymbol{u}}$ is given by

$$\begin{aligned} \boldsymbol{\mathcal{B}}^{-1}\,\dot{\boldsymbol{u}} &= \boldsymbol{\mathcal{B}}^{-1}\,(\frac{\mathrm{d}}{\mathrm{d}t}\,(\boldsymbol{\mathcal{B}}\,\underline{\boldsymbol{u}})) \\ &= \boldsymbol{\mathcal{B}}^{-1}\,(\boldsymbol{\mathcal{B}}\,\dot{\underline{\boldsymbol{u}}} + \dot{\boldsymbol{\mathcal{B}}}\,\underline{\boldsymbol{u}}) \\ &= \dot{\underline{\boldsymbol{u}}} + \boldsymbol{\mathcal{B}}^{-1}\,\dot{\boldsymbol{\mathcal{B}}}\,\underline{\boldsymbol{u}}\,. \end{aligned} \tag{35}$$

This is the general formula for differentiation in a moving coordinate system. If the coordinate system is stationary then $\dot{\boldsymbol{\mathcal{B}}} = \boldsymbol{0}$ and $\boldsymbol{\mathcal{B}}^{-1}\,\dot{\boldsymbol{u}} = \dot{\underline{\boldsymbol{u}}}$.

Expanding the term $\boldsymbol{\mathcal{B}}^{-1}\,\dot{\boldsymbol{\mathcal{B}}}$ gives

$$\begin{aligned} \boldsymbol{\mathcal{B}}^{-1}\,\dot{\boldsymbol{\mathcal{B}}} &= \boldsymbol{\mathcal{B}}^{-1}\begin{bmatrix}\dot{\boldsymbol{b}}_1 & \cdots & \dot{\boldsymbol{b}}_n\end{bmatrix} \\ &= \begin{bmatrix}\boldsymbol{\mathcal{B}}^{-1}\,\dot{\boldsymbol{b}}_1 & \cdots & \boldsymbol{\mathcal{B}}^{-1}\,\dot{\boldsymbol{b}}_n\end{bmatrix}, \end{aligned}$$

which is a square matrix whose columns are the coordinates of the derivatives of the basis vectors. Three special cases are of particular interest. If $\mathcal{C}$, $\mathcal{D}_O$ and $\mathcal{E}_O$ are the orthonormal and Plücker bases derived from a coordinate frame $Oxyz$ that is moving with a velocity of $\hat{\boldsymbol{v}}$ (coordinate vector $\hat{\underline{\boldsymbol{v}}}_O = [\,\underline{\boldsymbol{\omega}}^T\ \underline{\boldsymbol{v}}_O^T\,]^T$), then

$$\boldsymbol{\mathcal{C}}^{-1}\,\dot{\boldsymbol{\mathcal{C}}} = \underline{\boldsymbol{\omega}}\times\,, \tag{36}$$

$$\boldsymbol{\mathcal{D}}_O^{-1}\,\dot{\boldsymbol{\mathcal{D}}}_O = \hat{\underline{\boldsymbol{v}}}_O\times = \begin{bmatrix}\underline{\boldsymbol{\omega}}\times & \boldsymbol{0} \\ \underline{\boldsymbol{v}}_O\times & \underline{\boldsymbol{\omega}}\times\end{bmatrix} \tag{37}$$

and

$$\boldsymbol{\mathcal{E}}_O^{-1}\,\dot{\boldsymbol{\mathcal{E}}}_O = \hat{\underline{\boldsymbol{v}}}_O\times^* = \begin{bmatrix}\underline{\boldsymbol{\omega}}\times & \underline{\boldsymbol{v}}_O\times \\ \boldsymbol{0} & \underline{\boldsymbol{\omega}}\times\end{bmatrix}, \tag{38}$$

where $\underline{\boldsymbol{\omega}}\times$ is the 3×3 matrix that maps a 3D coordinate vector, $\underline{\boldsymbol{u}}$, to the cross product $\underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{u}}$, and $\hat{\underline{\boldsymbol{v}}}_O\times$ and $\hat{\underline{\boldsymbol{v}}}_O\times^*$ are the analogous spatial-vector operators. $\hat{\underline{\boldsymbol{v}}}_O\times$ maps a motion vector to a motion vector, while $\hat{\underline{\boldsymbol{v}}}_O\times^*$ maps a force to a force. $\underline{\boldsymbol{\omega}}\times$ is given by the formula

$$\begin{bmatrix}\omega_x \\ \omega_y \\ \omega_z\end{bmatrix}\times = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{39}$$

For the three special cases above, (35) becomes

$$\boldsymbol{\mathcal{C}}^{-1}\,\dot{\boldsymbol{v}} = \dot{\underline{\boldsymbol{v}}} + \underline{\boldsymbol{\omega}} \times \underline{\boldsymbol{v}}\,, \tag{40}$$

$$\boldsymbol{\mathcal{D}}_O^{-1}\,\dot{\hat{\boldsymbol{m}}} = \dot{\hat{\underline{\boldsymbol{m}}}} + \hat{\underline{\boldsymbol{v}}}_O \times \hat{\underline{\boldsymbol{m}}} \tag{41}$$

and

$$\boldsymbol{\mathcal{E}}_O^{-1}\,\dot{\hat{\boldsymbol{f}}} = \dot{\hat{\underline{\boldsymbol{f}}}} + \hat{\underline{\boldsymbol{v}}}_O \times^* \hat{\underline{\boldsymbol{f}}}\,, \tag{42}$$

where $\boldsymbol{v}$, $\hat{\boldsymbol{m}}$ and $\hat{\boldsymbol{f}}$ denote general elements of E^3, M^6 and F^6, respectively. Except for the use of basis-mapping notation, (40) is a standard result that can be found in many textbooks. Observe that we have been able to treat this subject using only one kind of differential operator, instead of the usual two. This is because the basis-mapping notation gives us a symbol for "the coordinate vector that represents…".

VII. Acceleration

Nothing has yet been said about the velocity of O. The role of O is to specify a point where something is measured, or a point through which something passes. Thus, quantities like $\boldsymbol{v}_O$, $\boldsymbol{n}_O$, $\boldsymbol{d}_{Ox}$, etc., all depend on the position of O, but not its velocity. It is therefore possible to assign any desired velocity to O without affecting the definitions of spatial vectors. However, if the velocity of O is nonzero, then $\mathcal{D}_O$ and $\mathcal{E}_O$ are moving bases, and this must be taken into account when differentiating a spatial vector.

Referring back to Figure 1, suppose we make O track a point in the moving body, so that the two points coincide permanently. The body's velocity would still be characterized by the two Euclidean vectors $\boldsymbol{\omega}$ and $\boldsymbol{v}_O$; it would still have a spatial velocity of $\hat{\boldsymbol{v}}$, as defined in (6); and $\hat{\boldsymbol{v}}$ would still be represented in $\mathcal{D}_O$ coordinates by its coordinate vector, $\hat{\underline{\boldsymbol{v}}}_O$, as defined in (7) and (8). However, O itself would now have a velocity of $\boldsymbol{v}_O$, and so would $Oxyz$.

Let $\hat{\boldsymbol{v}}_{Oxyz}$ denote the spatial velocity of the coordinate frame, and let $\hat{\underline{\boldsymbol{v}}}_{Oxyz}$ be the coordinate vector representing $\hat{\boldsymbol{v}}_{Oxyz}$ in $\mathcal{D}_O$ coordinates. If we set the angular velocity of the coordinate frame to zero, then $\hat{\underline{\boldsymbol{v}}}_{Oxyz} = [\,\underline{\boldsymbol{0}}^T\ \underline{\boldsymbol{v}}_O^T\,]^T$.

The acceleration of a rigid body is simply the time-derivative of its velocity. The spatial acceleration of the body in Figure 1 is therefore $\dot{\hat{v}}$, and the coordinate vector representing its acceleration in $\mathcal{D}_O$ coordinates is $\boldsymbol{\mathcal{D}}_O^{-1}\,\dot{\hat{v}}$. We can obtain an expression for $\boldsymbol{\mathcal{D}}_O^{-1}\,\dot{\hat{v}}$ directly from (41) as follows:

$$\begin{aligned}\boldsymbol{\mathcal{D}}_O^{-1}\,\dot{\hat{v}} &= \dot{\underline{\hat{v}}}_O + \underline{\hat{v}}_{Oxyz} \times \underline{\hat{v}}_O \\ &= \begin{bmatrix}\dot{\underline{\omega}} \\ \dot{\underline{v}}_O\end{bmatrix} + \begin{bmatrix}\underline{\mathbf{0}} \\ \underline{v}_O\end{bmatrix} \times \begin{bmatrix}\underline{\omega} \\ \underline{v}_O\end{bmatrix} \\ &= \begin{bmatrix}\dot{\underline{\omega}} \\ \dot{\underline{v}}_O\end{bmatrix} + \begin{bmatrix}\underline{\mathbf{0}} \\ \underline{v}_O \times \underline{\omega}\end{bmatrix}. \end{aligned} \tag{43}$$

In the classical textbook treatment, the acceleration of a rigid body is usually defined by an angular acceleration vector, $\dot{\omega}$, and the linear acceleration, $\ddot{r}$, of a chosen body-fixed point whose position is given by r. If we define r to be the position of O relative to some fixed datum, then $\underline{v}_O = \dot{\underline{r}}$ and $\dot{\underline{v}}_O = \ddot{\underline{r}}$, and (43) becomes

$$\boldsymbol{\mathcal{D}}_O^{-1}\,\dot{\hat{v}} = \begin{bmatrix}\dot{\underline{\omega}} \\ \ddot{\underline{r}}\end{bmatrix} + \begin{bmatrix}\underline{\mathbf{0}} \\ \dot{\underline{r}} \times \underline{\omega}\end{bmatrix}. \tag{44}$$

The first term on the right-hand side is the concatenation of the two vectors used in the classical description of rigid-body acceleration. For this reason, it is sometimes called the classical, or conventional, acceleration vector, so as to distinguish it from spatial acceleration. As can be seen from (43) and (44), the classical acceleration vector differs from the spatial acceleration by the term $[\,\underline{\mathbf{0}}^T\ (\underline{v}_O \times \underline{\omega})^T\,]^T$, which is attributable to the linear velocity of the frame $Oxyz$. Furthermore, the classical acceleration vector is the component-wise derivative of the spatial velocity vector in a Plücker coordinate system that is moving with a velocity of $[\,\underline{\mathbf{0}}^T\ \underline{v}_O^T\,]^T$. This is essentially the same result as reported in [5].

Without the aid of Plücker bases, it is possible to make the following erroneous argument: "The Euclidean vectors ω and $\dot{r}$ define the velocity of the rigid body; the coordinate vectors $\underline{\omega}$ and $\dot{\underline{r}}$ express ω and $\dot{r}$ in the basis $\mathcal{C} = \{i, j, k\}$, which is invariant (because $Oxyz$ is not rotating); therefore $\dot{\underline{\omega}}$ and $\ddot{\underline{r}}$ are the coordinate vectors that represent the derivatives of ω and $\dot{r}$; so $[\,\dot{\underline{\omega}}^T\ \ddot{\underline{r}}^T\,]^T$ is the coordinate vector representing the acceleration." The flaw in this argument becomes apparent as soon as we introduce the Plücker basis: the mapping from Plücker coordinates to the spatial velocity vector is defined by the basis $\mathcal{D}_O$, not $\mathcal{C}$, and if the velocity of O is not zero, then $\mathcal{D}_O$ is a time-varying coordinate system, and this must be taken into account when performing the differentiation.

VIII. Conclusion

This paper has introduced the concept of Plücker bases, and an operator notation to express explicitly how a basis maps a coordinate vector to the vector it represents. Using these tools, the paper explains the following: the precise relationship between spatial vectors and the pairs of 3-D vectors that define them; the correct way to differentiate a spatial vector in a moving Plücker coordinate system; and why the classical description of rigid-body acceleration is not the derivative of spatial velocity.

Although this paper has been written in the language of spatial vectors, the results reported here are applicable generally to any 6-D vector that is represented in Plücker coordinates.

References

[1] E. J. Baker and K. Wohlhart. Motor Calculus: A New Theoretical Device for Mechanics. Graz, Austria: Institute for Mechanics, TU Graz, 1996. (English translation of [8] and [9].)

[2] R. S. Ball. *A Treatise on the Theory of Screws*. London: Cambridge University Press, 1900. Republished 1998.

[3] L. Brand. *Vector and Tensor Analysis*, 4th ed. New York/London: Wiley/Chapman and Hall, 1953.

[4] R. Featherstone. *Robot Dynamics Algorithms*. Boston: Kluwer Academic Publishers, 1987.

[5] R. Featherstone. The Acceleration Vector of a Rigid Body. *Int. J. Robotics Research*, 20(11):841–846, 2001.

[6] O. Khatib. Inertial Properties in Robotic Manipulation: An Object-Level Framework. *Int. J. Robotics Research*, 14(1):19–36, 1995.

[7] H. Lipkin. Time Derivatives of Screws with Applications to Dynamics and Stiffness. *Mechanism and Machine Theory*, 40(3):259–273, 2005.

[8] R. von Mises. Motorrechnung, ein neues Hilfsmittel der Mechanik [Motor Calculus: a New Theoretical Device for Mechanics]. *Zeitschrift für Angewandte Mathematik und Mechanik*, 4(2):155–181, 1924.

[9] R. von Mises. Anwendungen der Motorrechnung [Applications of the Motor Calculus]. *Zeitschrift für Angewandte Mathematik und Mechanik*, 4(3):193–213, 1924.

[10] R. M. Murray, Z. Li and S. S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. Boca Raton, FL: CRC Press, 1994.

[11] J. Plücker. Fundamental Views Regarding Mechanics. *Philosophical Transactions*, 156:361–380, 1866.

[12] J. M. Selig. *Geometrical Methods in Robotics*. New York: Springer, 1996.

[13] S. Stramigioli and H. Bruyninckx. Geometry of Dynamic and Higher-Order Kinematic Screws. Proc. IEEE Int. Conf. Robotics and Automation, Seoul, Korea, pp. 3344–3349, 2001.

[14] K. Wohlhart. Time Derivatives of the Velocity Motor. Proc. SYROM 2001, Bucharest, Romania, vol. 2, pp. 371–376, 2001.

[15] K. Wohlhart. Mises Derivatives of Motors and Motor Tensors. Proc. RoManSy 2002, Udine, Italy, pp. 87–98, 2002.

[16] L. S. Woo and F. Freudenstein. Application of Line Geometry to Theoretical Kinematics and the Kinematic Analysis of Mechanical Systems. *Journal of Mechanisms*, 5:417–460, 1970.