Angular Momentum Based Controller for Balancing an Inverted Double Pendulum

Morteza Azad[*] and Roy Featherstone[*]
[*] School of Engineering, Australian National University, Canberra, Australia

Abstract. This paper presents a new control algorithm, based on angular momentum, for balancing a planar inverted double pendulum robot having one degree of underactuation. The robot may either pivot about a fixed point, or roll with a curved foot over a flat ground. The controller is able to stabilize the robot in any unstable balanced configuration, and to follow arbitrary motion trajectories without losing balance. The latter necessarily involves some tracking error. Several simulation results are presented.

1 Introduction

This paper considers the problem of balancing a planar inverted double pendulum (IDP) consisting of two bodies connected by an actuated revolute joint. The lower body contains a curve, called the foot, which makes a rolling contact with a flat supporting surface (the ground). If the curve shrinks to a single point then the rolling contact simplifies to a passive revolute joint. In this case, the IDP resembles the acrobot (for acrobatic robot) which was introduced by Hauser and Murray (1990). The control problem for the acrobot has been studied by several researchers. The pioneering work of Spong (1995) used a LQR controller for balancing the acrobot in its upright equilibrium position. Xin and Kaneda (2001), Inoue et al. (2007) and Lai et al. (2005) also used the same controller for balancing the acrobot.

Berkemeier and Fearing (1999) introduced a controller based on zero dynamics for trajectory tracking of the acrobot. They found a class of interesting feasible trajectories for the acrobot and achieved theoretically accurate trajectory tracking performance using their proposed controller.

Yamakita et al. (2002), and Yonemura and Yamakita (2004) proposed an output zeroing controller for balancing the acrobot. Their proposed controller uses an output function which is defined by the angular momentum and one other new state. To work out the value of the torque, they had to calculate the third derivative of the angular momentum.

Grizzle et al. (2005) considered the general case and designed a nonlinear controller for mechanical systems with one degree of underactuation. Their output function becomes the same as that proposed by Yamakita et al. (2002) and Yonemura and Yamakita (2004) for the acrobot case. This approach also needs to work out the third derivative of the angular momentum.

In this paper we introduce a new simple controller for an IDP mechanism which is based on its angular momentum about the contact point between the robot and the ground. The new controller is able to follow setpoint commands, in which only the target configuration is given, and also motion trajectory commands, in which the motion of the actuated joint is a prescribed function of time. However, the latter necessarily involves significant tracking errors for the purpose of maintaining balance. The controller works with both a point foot and a curved foot.

2 Model of an IDP

We consider an IDP consisting of two links connected by an actuated revolute joint. The lower link contains a surface called the foot, which makes contact with the ground at a single point. Three foot shapes are considered, as shown in Figure 1: a single point, a circular arc and a general convex curve. In the first case, the contact can be modelled as a revolute joint. In the other cases, it must be modelled as a rolling-contact joint. In all three cases, the joint has a single joint variable, q_1, which is an angle. We assume that the ground is flat and horizontal, and that there is no slipping or loss of contact betwen the foot and the ground.

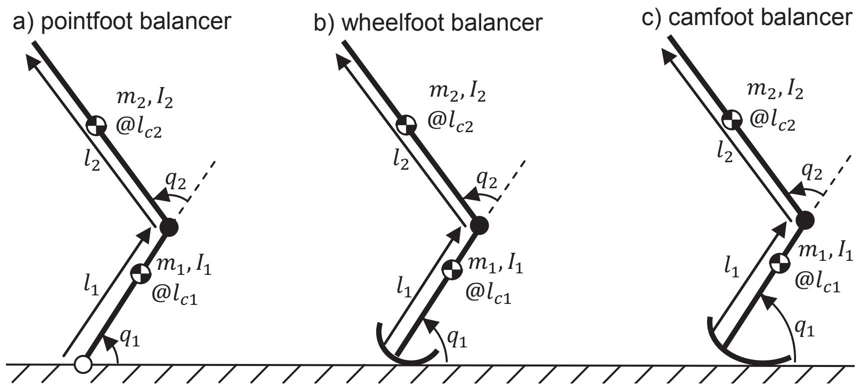

Figure 1. Robot models and parameters

3 Controlling Angular Momentum

In this section, we develop a control law for the case of a pointfoot balancer. Let X_G denote the horizontal displacement of the centre of mass (CoM) relative to the contact point, and let L denote the angular momentum of the robot about the contact point. The conditions for balance are: $X_G = 0$ and $\dot{q}_1 = \dot{q}_2 = 0$. However, as $\dot{X}_G$ and L are both linear functions of $\dot{q}_1$ and $\dot{q}_2$, the two velocity constraints can be replaced with $\dot{X}_G = 0$ and $L = 0$. Now, the rate of change of angular momentum of a multibody system about any fixed point equals the moment about that point of the external forces acting on the system. As we have chosen to express L at the contact point, which is the one point about which the moment of the ground reaction force is always zero, it follows that $\dot{L}$ must equal the moment of the gravitational force about the contact point. So we have

$$\dot{L} = -(m_1 + m_2)gX_G \,, \tag{1}$$

(where g is gravitational acceleration) and therefore also

$$\ddot{L} = -(m_1 + m_2)g\dot{X}_G \,. \tag{2}$$

The conditions for balance can now be written $L = \dot{L} = \ddot{L} = 0$; so we consider the angular momentum as an output function and define a feedback controller that drives the output function to zero exponentially. So the **control law** would be

$$\tau = k_{dd}\ddot{L} + k_d\dot{L} + k_pL + \tau^d \,, \tag{3}$$

where k_{dd}, k_d and k_p are controller gains, and τ^d is the correct holding torque at the actuated joint for the desired balanced configuration. The effect of τ^d is to make this configuration an equilibrium point of the closed-loop system. If q_1^d and q_2^d are the joint angles in the desired configuration, then

$$\tau^d = m_2 l_{c2} g \cos(q_1 + q_2) \,. \tag{4}$$

This controller is equivalent to

$$\tau = -k_v\dot{X}_G - k_xX_G + k_pL + \tau^d \tag{5}$$

where $k_v = (m_1 + m_2)gk_{dd}$ and $k_x = (m_1 + m_2)gk_d$. We use this alternative control law for balancers that make a rolling contact with the ground. Note that these control laws are not robust to modelling errors: if τ^d is not exactly the correct holding torque then the controller will not converge exactly to the desired configuration, but to a nearby one instead.

3.1 Stability Analysis and Gain Calculation

Considering the nonlinear state-space equations for an IDP as

$$\dot{x} = f(x) + g(x) \, . \, u \tag{6}$$

where $x = (q_1 - q_1^d, q_2 - q_2^d, \dot{q}_1, \dot{q}_2)$, and knowing that $u = \tau$ is a function of x, we will have $\dot{x} = h(x)$. Linearizing about $x = 0$ gives

$$\dot{x} = A\,x \tag{7}$$

where $A = \frac{\partial h}{\partial x}|_{x=0}$ is a 4×4 matrix. To check the stability of the system and calculate the controller gains, first we need to calculate the eigenvalues of matrix A, which are the roots of its characteristic equation. The characteristic equation for the pointfoot balancer has the general form:

$$\lambda^4 + (b_1 k_{dd} - b_2 k_p)\lambda^3 + (b_3 k_d - \alpha)\lambda^2 + (b_4 k_p)\lambda + a = 0 \tag{8}$$

where b_i, α and a are parameters which are dependent on matrix A and the desired configuration. This system is stable if all four eigenvalues are negative. Since a is always positive, one possible solution for Equation 8 would be

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = -p \tag{9}$$

so

$$(\lambda + p)^4 = \lambda^4 + 4p\lambda^3 + 6p^2\lambda^2 + 4p^3\lambda + p^4 = 0 \tag{10}$$

By this assumption, all poles of the closed loop system are negative ($p = \sqrt[4]{a}$) so the system is always stable and the controller gains become

$$k_p = \frac{4\,p^3}{b_4}\,, \qquad k_d = \frac{6\,p^2 + \alpha}{b_3}\,, \qquad k_{dd} = \frac{4\,p + b_2\,k_p}{b_1}\,. \tag{11}$$

4 Rolling Contact

Rolling contact involves a moving contact point. If $R(q_1)$ expresses the radius of curvature of the foot at the contact point as a function of the joint angle q_1, then the velocity of the contact point is $-R(q_1)\dot{q}_1$. For the pointfoot balancer, $R(q_1) = 0$ for all q_1; for the wheelfoot balancer, $R(q_1)$ is a positive constant; and for the camfoot balancer, we have only that $R(q_1) \geq 0$ for all q_1, which is the general case. Note that $R(q_1)$ is, in general, only a piecewise-continuous function of q_1. If $R(q_1)$ changes discontinuously at some value of q_1 then the velocity of the contact point also changes discontinuously.

To balance on a rolling contact, we use the control law in Equation 5. As X_G is defined to be the horizontal position of the CoM relative to the contact point, it follows that $\dot{X}_G$ is the horizontal velocity of the CoM relative to the contact point; so $\dot{X}_G$ is capable of changing discontinuously. The method of calculating gains is the same as for the pointfoot balancer, although Equation 8 now contains a few small-magnitude terms that are nonlinear in the gains. We have ignored these terms when calculating the gains.

5 Following a Trajectory

To make the robot move from one balanced configuration to another, one simply changes τ^d to the correct value for the new configuration. However, to make the robot follow a prescribed trajectory is harder. We define a motion trajectory for an IDP to be an equation that specifies q_2 as an explicit function of time. In general, following such a trajectory will be physically impossible without losing balance. Therefore, we expect the controller to follow the trajectory as nearly as possible subject to the constraint that it must maintain the robot's balance. This implies significant trajectory tracking errors.

To implement trajectory following, we use the control law in Equation 3, but replace L with $(L-L^d)$ and calculate τ^d differently. L^d is the theoretical value of L assuming that the robot happens to be perfectly balanced at the current instant and following the trajectory exactly; and τ^d is likewise the theoretical value of τ under these same assumptions.

6 Simulation Results

The parameters that we have used in our simulations are shown in Table 1. The radius of the foot of the wheelfoot balancer is 5cm. The foot of the camfoot balancer is composed of a clothoid curve segment and its mirror image, the two being joined to form a shield shape with a sharp point at the bottom. The radius of curvature of the clothoid is 0.15 immediately adjacent to the sharp point, and reduces gradually with increasing distance from the point. In an upright configuration, camfoot is pivotting on its sharp point. In a crouched configuration, it is rolling on the clothoid.

6.1 Balancing in Upright and Crouched Configurations

Figures 2(a), 2(c) and 2(e) show the pointfoot, wheelfoot and camfoot balancers moving to an upright balanced position ($q_2 = 0$) from a crouched position ($q_2 = -\frac{\pi}{2}$). In each case, the initial velocity is zero and the initial value of q_1 is calculated for perfect balance given $q_2 = -\frac{\pi}{2}$. Figures 2(b), 2(d) and 2(f) show the corresponding crouching motions, in which each balancer starts in a balanced upright position ($q_2 = 0$) and moves to a balanced crouching position ($q_2 = -\frac{\pi}{2}$). The gains used for these six motions are shown in Table 2. In every case, the gains are computed by linearizing about the target position using the method described in section 3.1. The curves for the camfoot balancer show sharp changes at about $t = 0.5$ in Figure 2(e), and $t = 1$ in Figure 2(f). These are caused by the large step change in $R(q_1)$ as the balancer transitions between pivotting and rolling.

6.2 Trajectory Tracking

Figure 2(g) shows the pointfoot balancer following a trajectory comprising two steps (including one at $t = 0$), one ramp and a sine wave. The gains used in this simulation are computed by linearizing about the target position for step and ramp commands, and the midpoint position for the sine wave command. The graph shows significant tracking errors. However this is only to be expected, as the command is physically impossible to follow. Therefore, the control system must find some physically possible trajectory to follow instead. It is not known what fraction of the tracking error is physically necessary, and what fraction is due to the controller being less than perfect.

Table 1. Simulation parameters (SI units)

robots	m_1	m_2	l_1	l_{c1}	l_2	l_{c2}	I_1	I_2
pointfoot	7	7	0.5	0.5	0.75	0.75	0	0
wheelfoot & camfoot	0.5	1	0.3	0.15	0.4	0.2	$\frac{1}{12}m_1l_1^2$	$\frac{1}{4}m_2l_2^2$

Table 2. Gains

	balancing			crouching to $q_2^d = -\frac{\pi}{2}$		
robots	k_{dd}	k_d	k_p	k_{dd}	k_d	k_p
pointfoot	0.9704	10.2615	21.9843	1.698	11.4853	18.8457
wheelfoot	2.6545	16.3137	27.1036	1.8542	12.4549	21.0603
camfoot	0.625	8.4534	24.2363	2.093	13.0462	20.1408

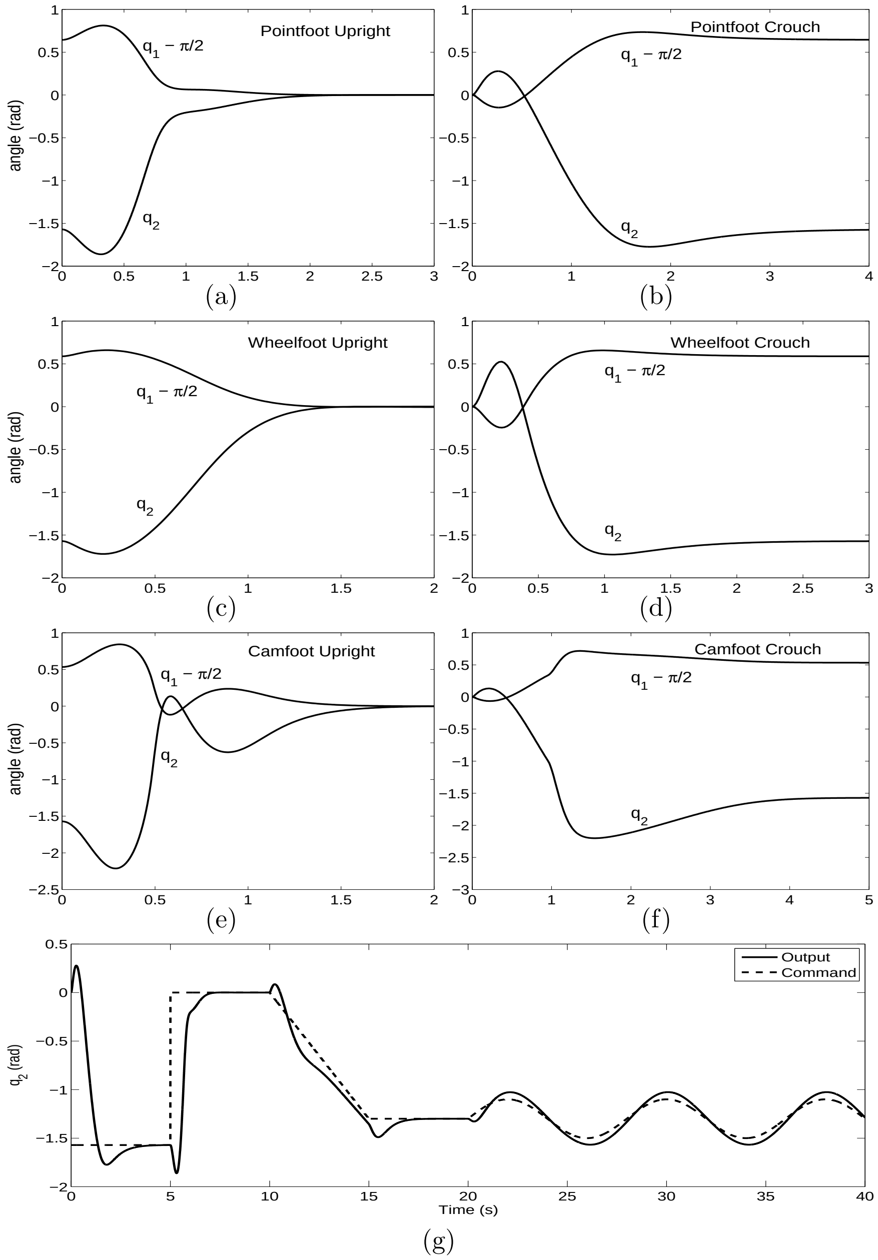

Figure 2. Simulation results for balancing upright (a, c, e), crouching (b, d, f) and trajectory tracking (g)

7 Conclusions and Future Work

In this paper we have described a new nonlinear angular momentum based balancing controller for an IDP. Simulation results show very good performance of the controller in stabilizing the IDP at unstable balanced configurations with both point and rolling contact between the robot's foot and the ground. Simulation results of tracking an arbitrary trajectory (combination of step, ramp and sine wave) for the pointfoot balancer show significant tracking errors, but some amount of tracking error is physically necessary in order to maintain balance. This work is part of a project that aims ultimately to create a machine with only two actuators that can hop and balance in 3D.

Bibliography

M. D. Berkemeier and R. S. Fearing. Tracking fast inverted trajectories of the underactuated acrobot. *IEEE Trans. Robotics and Automation*, 15(4):740–750, August 1999.

J. W. Grizzle, C. H. Moog, and C. Chevallereau. Nonlinear control of mechanical systems with an unactuated cyclic variable. *IEEE Trans. Automatic Control*, 50(5):559–576, May 2005.

J. Hauser and R. M. Murray. Nonlinear controllers for nonintegrable systems: the acrobot example. In *Proc. American Control Conf.*, pages 669–671, San Diego, CA, 23–25 May 1990.

A. Inoue, M. Deng, S. Hara, and T. Henmi. Swing-up and stabilizing control system design for an acrobot. In *Proc. IEEE Int. Conf. Networking, Sensing and Control*, pages 559–561, London, UK, 15–17 April 2007.

X. Lai, Y. Wu, J.She, and M. Wu. Control design and comprehensive stability analysis of acrobots based on lyapunov functions. *J. Central South University of Technology*, 12(1):210–216, 2005.

M. W. Spong. The swing up control problem for the acrobot. *IEEE Control Systems*, 15(1):49–55, 1995.

X. Xin and M. Kaneda. A new solution to the swing up control problem for the acrobot. In *Proc. 40th SICE Annual Conf.*, pages 124–129, Nagoya, Japan, 25–27 July 2001.

M. Yamakita, T. Yonemura, Y. Michitsuji, and Z. Luo. Stabilization of acrobot robot in upright position on a horizontal bar. In *Proc. IEEE Int. Conf. Robotics and Automation*, pages 3093–3098, Washington, DC, 11–15 May 2002.

T. Yonemura and M. Yamakita. Swing up control problem of acrobot based on switched output functions. In *Proc. SICE Annual Conf.*, pages 1909–1914, Sapporo, 4–6 August 2004.