Analysis and Design of Planar Self-Balancing Double-Pendulum Robots

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Abstract This paper explores the attributes of a double-pendulum robot that determine its ability to balance. A new measure is defined, called the velocity gain, that expresses the degree to which the robot's centre of mass will move in response to motion of the robot's actuated joint. This measure can be used both to analyse a robot's performance and to design robot mechanisms for improved performance. Some properties of the velocity gain are explained, and several examples of both good and bad balancing robots are presented. The significance of this work is that a robot mechanism's intrinsic ability to balance sets a physical upper limit to the robot's attainable performance at balancing tasks, which is independent of the choice of control system.

1 Introduction

Balancing is an important activity for some kinds of mobile robot. A great deal of research has been devoted to the development of control systems for balancing, of which Grizzle et al. (2005); Spong (1995); Xinjilefu et al. (2009) is just a small sample; but there appears to have been no corresponding effort to study the physical attributes of a robot mechanism—its kinematic and inertia parameters—that govern its ability to perform balancing tasks.

This paper presents a study of balancing in 2D performed by a planar double-pendulum robot making a single point contact with a supporting surface, this being the simplest case of a self-balancing mobile robot. As the shape of the foot is an important factor, this paper considers the case of a general convex curve that may contain sharp points. Thus, both the case of balancing on a sharp point and balancing on a rolling contact will be considered.

To achieve and maintain balance, the control system must adjust the robot's centre of mass (CoM) indirectly via motions of the actuated joint.

The overall performance will depend on the quality of the sensors, the effectiveness of the control system, and the degree to which the CoM is affected by motions of the actuated joint. This last item is the subject of this paper: it is a property of the mechanism itself, and it sets a physical upper limit to the achievable performance of the robot.

This paper introduces a dimensionless, quantitative measure of the balancing ability of a double-pendulum robot mechanism, called the *velocity gain*. It expresses the ratio between a velocity change at the actuated joint and the resulting velocity change of the CoM. The measure is defined; a method for calculating it is described; a few basic properties are stated; and a formula is presented for the special case of a sharp pointed foot. Then the issue of parameter reduction is discussed, and how parameters can be varied without changing the velocity gain. Finally, several examples of double-pendulum mechanisms are presented, including both good and bad balancers, and balancers with unusual properties; and some general comments are made on the design of good balancers.

2 Velocity Gain

Figure 1(a) shows a planar double pendulum consisting of a leg (link 1) connected to a torso (link 2) via an actuated revolute joint (joint 2). The leg makes a single-point rolling contact with the ground, which we shall call joint 1. We assume that there is no slipping or loss of contact between the leg and the ground. Joint 1 allows a single degree of motion freedom, and can be characterized by a single joint variable, q_1, which is an angle, plus a description of the shape of the foot. (We shall assume that the ground

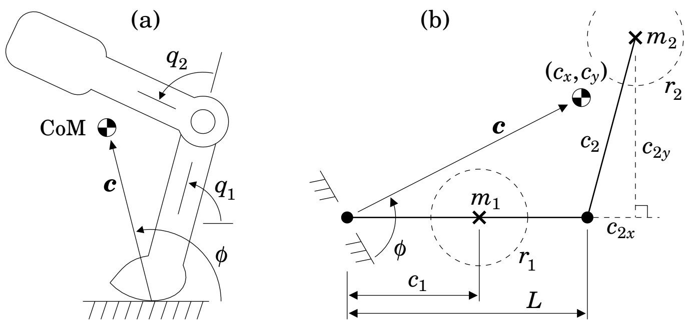

Figure 1. Definition of velocity gain (a), and parameters of a double pendulum modelled as a planar 2R mechanism (b).

is flat and horizontal.) If the contact point coincides with a sharp point on the foot, then joint 1 simplifies to a revolute joint for some range of joint angles that depends on the exact shape of the foot.

Figure 1(a) also shows a vector $\boldsymbol{c}$, which locates the robot's CoM relative to the contact point, and an angle ϕ, which gives the direction of $\boldsymbol{c}$. ϕ has been shown measured relative to the horizontal, but it could be measured relative to any desired fixed direction.

We seek a performance measure for the balancing ability of this robot. Now, the task of balancing requires control of the angle ϕ, but the only quantity that the control system can control directly is q_2. Therefore, the ability of this robot to balance depends critically on the degree to which changes in q_2 can cause changes in ϕ.

As this property will vary with configuration, let us define the following quantity as a measure of a double pendulum's balancing performance locally at configuration $\boldsymbol{q} = [q_1\ q_2]^{\mathrm{T}}$:

$$G_{\mathrm{v}}(\boldsymbol{q}) = \lim_{\Delta q_2 \to 0} \frac{\Delta\phi}{\Delta q_2} = \frac{\Delta\dot{\phi}}{\Delta\dot{q}_2}\,, \tag{1}$$

where $\Delta\dot{\phi}$ and $\Delta\dot{q}_2$ are the instantaneous changes in $\dot{\phi}$ and $\dot{q}_2$ caused by an impulsive torque applied at joint 2 with the mechanism in configuration $\boldsymbol{q}$. The magnitude of the impulse is unimportant, as it does not affect the ratio. G_{v} can also be defined as the ratio of two accelerations: $G_{\mathrm{v}}(\boldsymbol{q}) = \Delta\ddot{\phi}/\Delta\ddot{q}_2$ where $\Delta\ddot{\phi}$ and $\Delta\ddot{q}_2$ are the instantaneous changes in $\ddot{\phi}$ and $\ddot{q}_2$ due to a step change in applied torque at joint 2. Nevertheless, we shall call G_{v} the *velocity gain* of the mechanism. A double pendulum is a good balancer if $|G_{\mathrm{v}}|$ is sufficiently large at every configuration where balancing is required to take place.

Calculation Method. To calculate $G_{\mathrm{v}}(\boldsymbol{q})$, the first step is to calculate the joint-space inertia matrix at $\boldsymbol{q}$, which we shall call $\boldsymbol{H}$, and use it to work out the step response to a nonzero impulse at joint 2:

$$\begin{bmatrix}\Delta\dot{q}_1\\ \Delta\dot{q}_2\end{bmatrix} = \boldsymbol{H}^{-1}\begin{bmatrix}0\\ u\end{bmatrix}, \tag{2}$$

where $u \neq 0$ is the impulse magnitude. This equation can be solved for $\Delta\dot{q}_1$, giving

$$\Delta\dot{q}_1 = \frac{-H_{12}}{H_{11}}\Delta\dot{q}_2\,. \tag{3}$$

The next step is to calculate $\boldsymbol{c}$ and $\Delta\dot{\boldsymbol{c}}$. The latter depends on $\boldsymbol{q}$ and $\Delta\dot{q}_2$ only, as $\Delta\dot{q}_1$ can be eliminated using Eq. 3. (Bear in mind that $\dot{\boldsymbol{c}}$ gives the relative velocity of the CoM with respect to the moving contact point.) Finally, we calculate $G_\mathrm{v}(\boldsymbol{q})$ from

$$G_\mathrm{v}(\boldsymbol{q}) = \frac{\Delta\dot{\phi}}{\Delta\dot{q}_2} = \frac{\boldsymbol{b}\cdot\Delta\dot{\boldsymbol{c}}}{|\boldsymbol{c}|\,\Delta\dot{q}_2}\,, \tag{4}$$

where $\boldsymbol{b}$ is a unit vector at right-angles to $\boldsymbol{c}$ in the direction of increasing ϕ.

A Special Case. If the robot is pivotting on a sharp point, so that joint 1 is effectively a revolute joint, then G_v depends only on q_2, and is given by the following relatively simple formula (which takes approximately one page of algebra to derive):

$$G_\mathrm{v}(q_2) = \frac{c_y^2 + \frac{m_2\,c_x c_{2x}}{m_1+m_2}}{c_x^2 + c_y^2} - \frac{H_{12}}{H_{11}} \tag{5}$$

where

$$\begin{aligned}
H_{11} &= m_1(r_1^2 + c_1^2) + m_2(r_2^2 + (L + c_{2x})^2 + c_{2y}^2) \\
H_{12} &= m_2(r_2^2 + c_{2x}(L + c_{2x}) + c_{2y}^2) \\
c_x &= (m_1 c_1 + m_2(L + c_{2x}))/(m_1 + m_2) \\
c_y &= m_2 c_{2y}/(m_1 + m_2) \\
c_{2x} &= c_2\cos(q_2) \qquad \text{and} \qquad c_{2y} = c_2\sin(q_2)\,.
\end{aligned}$$

The seven parameters m_i, c_i, r_i and L define the kinematic and inertia properties of the mechanism, and are shown in Figure 1(b). r_i is the radius of gyration of link i: if I_i is the rotational inertia of link i about its CoM then $I_i = m_i r_i^2$.

Properties. G_v is defined everywhere except where $\boldsymbol{c} = \boldsymbol{0}$ and values of q_1 where there is a step change in foot curvature at the contact point. (The latter causes a step change in G_v, making it only a piecewise-continuous function of q_1.) The definition of G_v as the ratio of two angular velocities makes it a dimensionless quantity. This implies that G_v is invariant with respect to a uniform scaling of the mass and/or the linear scale of the mechanism. In general, G_v depends on both q_1 and q_2, but it becomes independent of q_1 if joint 1 is revolute, and independent of q_2 if $c_2 = 0$. G_v can be positive, negative or zero, and its sign can vary with configuration. Robust balancing is impossible at or near a zero-crossing of G_v.

Parameter Reduction. Whether the objective is to design or to analyse a robotic balancer, it is useful to identify a minimal set of parameters that determine the value of G_v. The remaining parameters are then redundant in the sense that they can be varied without affecting G_v. Once these parameters are identified, one can use the minimal set to search for a mechanism with a desired property, and the redundant set to generate a family of other mechanisms with the same property.

For example, the fact that G_v is mass- and scale-invariant allows us to set the overall mass and size of the mechanism being analysed or designed to any convenient fixed value (e.g. $L = 1$ and $m_1 + m_2 = 1$ in Figure 1(b)) without loss of generality. Having found a single mechanism with an interesting property, one can immediately generate a two-parameter family of mechanisms with this same property by scaling the mass and size.

In addition to these scalings, there are potentially two more parameter reductions available. As explained in Featherstone (2008, §9.7), whenever two bodies are connected by a revolute joint, it is possible to add a particle of mass m to one of these bodies, and a particle of mass $-m$ to the other, without altering the dynamics of the mechanism. The only condition is that the two particles must coincide permanently, which implies that they must both lie on the joint's rotation axis. Applying this idea at joint 2 allows one of the remaining inertia parameters to be fixed; and it can be applied also at joint 1, if it is revolute, allowing one more parameter to be fixed. (Something similar is possible at joint 1 if the foot is a circular arc.)

To give a concrete example, if $c_2 > r_2$ in Figure 1(b) then it is possible to add a negative mass m satisfying $0 > m > -m_2$ to link 2 at joint 2 such that the modified link has $r_2 = 0$ (i.e., it has become a point mass). Repeating the process at joint 1 allows us to set $r_1 = 0$. Thus, without loss of generality, we can restrict our attention to mechanisms satisfying $r_1 = r_2 = 0$, $L = 1$ and $m_1 + m_2 = 1$, which implies that $G_v(q_2)$ is in fact a function of only three independent parameters. Having found a mechanism of interest, one can immediately generate a four-parameter family of mechanisms with the same property. The zero-gain example in the next section was obtained in this way.

3 Examples

Table 1 shows the inertia and kinematic parameters of several double-pendulum mechanisms having a revolute joint 1; and Figure 2 plots their velocity gains against q_2. The parameters are as defined in Figure 1, except I_i which is defined as $I_i = m_i r_i^2$. These mechanisms are all symmetrical with respect to q_2, so it is sufficient to plot G_v in the range 0 to π.

The first example consists of two identical thin rods having unit mass, unit length and uniform mass distribution. This example has a gain of

Table 1. Parameters of several balancing mechanisms.

mechanism	m_1	c_1	I_1	m_2	c_2	I_2	L
two rods	1	0.5	0.0833	1	0.5	0.0833	1
acrobot	7	0.5	0	7	0.75	0	0.5
good balancer	0.49	0.1714	0.0036	0.11	0.4364	0.0043	0.4
zero gain	0.5	0.2	0.26	0.8	1.125	0.0675	1
zero crosser	2	0.5	0	7	0.75	0	0.5

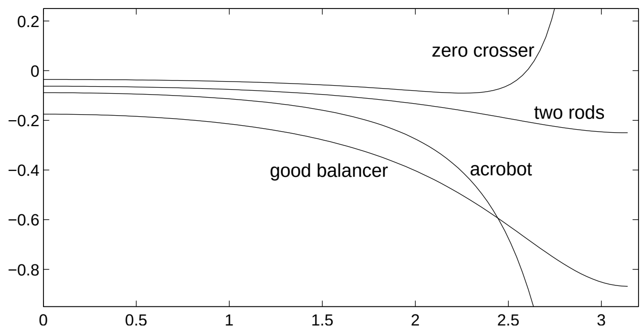

Figure 2. Velocity gain vs. q_2 for the mechanisms in Table 1.

$-1/16$ at $q_2 = 0$, increasing in magnitude to $-1/4$ at $q_2 = \pi$. That is a relatively low-magnitude gain, making this a poor balancer. Starting from an upright position, this mechanism would have to rotate q_2 by at least 16° in order to correct a 1° error in ϕ.

The next example is the acrobot as described in Berkemeier and Fearing (1998). (There is more than one published parameter set for this robot.) This mechanism is a significantly better balancer, with $G_\mathrm{v}(0) = -0.08867$ and $G_\mathrm{v}(\pi) = -3.6$. It is very common for $|G_\mathrm{v}|$ to be larger at π than at 0. This is partly because the CoM is closer to the foot, implying that $|\boldsymbol{c}|$ in Eq. 4 is smaller.

The next example is the result of an optimization process: starting with a mechanism consisting of a 0.4m rod weighing 0.04kg for the leg, a 0.6m rod weighing 0.06kg plus a 0.05kg point mass at the tip for the torso, and a 0.05kg point mass at joint 2 representing the joint bearing and drive, the optimization task was to find the optimal location, rounded to the nearest cm, of a 0.4kg point mass (representing an actuator) to be added to the leg. The objective function was $\int_0^{2\pi/3} G_\mathrm{v}(x)\,\mathrm{d}x$, and the optimal location was found to be 0.14m from the foot. This is a good balancer, having

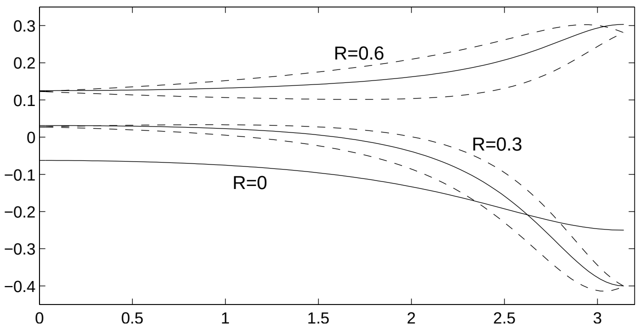

Figure 3. Effect of foot curvature on velocity gain.

$G_v(0) = -0.175$. Generally speaking, $|G_v(0)|$ will be large for mechanisms with a long, light torso and a short, heavy leg, the masses being concentrated at the tip of the torso and towards the bottom of the leg.

The next example has a velocity gain of zero everywhere, and is therefore impossible to balance. It was obtained by starting with the parameter set $m_1 = c_1 = 0$, $I_1 = I_2 = 0.2$ (value not critical), $m_2 = c_2 = L = 1$, which has the desired zero-gain property but specifies physically-impossible inertias, and then manipulating it in ways that do not affect G_v (as explained earlier) in order to find a parameter set with physically-possible inertias.

The final example shows G_v starting negative, but crossing zero as q_2 approaches π, ending up with $G_v(\pi) = 5.6$. Despite the high gain at π, this is a poor balancer at $|q_2| < 2.6$. Typically, we see this pattern when joint 1 lies inside the orbit of the CoM. This example was obtained from the acrobot parameter set by reducing the mass at joint 2 until the orbit of the CoM comfortably included joint 1.

Having considered several examples where the robot is pivotting on a sharp point, so that joint 1 is effectively a revolute joint, let us now consider the case of a circular foot making rolling contact with the ground, in which case joint 1 is a rack-and-pinion joint. The value of G_v for a general curved foot is instantaneously identical to that for a circular foot matching the tangent and radius of curvature of the general foot at the contact point.

Figure 3 plots G_v against q_2 for a modified version of the two-rods example in Table 1, in which the foot is a circle of radius R centred on the point $(R, 0)$ in the coordinate system of Figure 1(b). As G_v now depends on q_1 as well as q_2, three curves are plotted for each value of R: one in which q_1 has been calculated to put the mechanism in its unstable balanced configuration for each given value of q_2 (solid line), and two in which q_1 is offset by 30° each side of unstable balance (dashed lines).

Curves are plotted for $R = 0$, $R = 0.3$ and $R = 0.6$. The first is just a repeat of the two-rods curve in Figure 2, while the others show a general pattern of G_v becoming more positive as R increases. As a result, $R = 0.3$ makes for a very poor balancer, but $R = 0.6$ is almost twice as good as $R = 0$. For this mechanism, there is a qualitative change at $R = 0.5$, which is the point where the circle centre crosses inside the orbit of the CoM. Another qualitative change occurs at $R = 1$, which is where configuration $\boldsymbol{q} = \mathbf{0}$ switches from unstable to stable balance. Systems with $0.5 < R < 1$ may be able to exploit both stable and unstable balance, and switch between them, if the controller knows what to do.

4 Conclusion

This paper has presented a dimensionless measure, called the velocity gain, which quantifies the ability of a planar double-pendulum robot to perform balancing tasks. It can be used to analyse a given mechanism, or to design a mechanism that achieves a desired level of performance. Several examples are presented showing how the velocity gain is affected by the mechanism's kinematic and inertia parameters and the shape of the foot. The significance of this work is that velocity gain sets a physical upper limit to a robot's balancing ability, which is independent of the choice of control system. The concept of velocity gain can be extended to 3D.

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