Answers

for *Spatial Vector Algebra*
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Question A1

(a) $\begin{bmatrix} 0 \\ \cos(\theta) \\ \sin(\theta) \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cos(\theta) \\ \sin(\theta) \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ 0 \\ 2 \\ 4 \\ -2 \\ 1 \end{bmatrix}$

Question A2

(a) $\boldsymbol{d}_x$, $\boldsymbol{d}_y$ and $\boldsymbol{d}_z$ are the same in both bases because these vectors depend only on the x, y and z directions, which are the same for both coordinate frames. We also have $\boldsymbol{d}_{Qy} = \boldsymbol{d}_{Oy}$ because $Qy = Oy$. Thus, the only two vectors that are different in D_Q are

$$\boldsymbol{d}_{Qx} = \boldsymbol{d}_{Ox} - l\,\boldsymbol{d}_z \qquad \text{and} \qquad \boldsymbol{d}_{Qz} = \boldsymbol{d}_{Oz} + l\,\boldsymbol{d}_x\,.$$

Tip: A quick way to work out the answer is to imagine a rigid body performing the rotation you want to represent, and ask what happens to the body-fixed point at O. For example, if the body performs a rotation about Qx at unit angular velocity then the body-fixed point at O will move straight down with a linear velocity magnitude of l, so $\boldsymbol{d}_{Qx} = \boldsymbol{d}_{Ox} - l\,\boldsymbol{d}_z$.

(b) The coordinates ω_x, ω_y and ω_z are the same in both vectors. To obtain expressions for the linear coordinates, we use the formula $\boldsymbol{v}_Q = \boldsymbol{v}_O - \overrightarrow{OQ} \times \boldsymbol{\omega}$ with $\overrightarrow{OQ} = [0\ l\ 0]^{\mathrm{T}}$. This gives

$$\begin{aligned} v_{Qx} &= v_{Ox} - l\,\omega_z \\ v_{Qy} &= v_{Oy} \\ v_{Qz} &= v_{Oz} + l\,\omega_x \end{aligned}$$

(c) $$\begin{aligned} &\omega_x \boldsymbol{d}_{Qx} + \omega_y \boldsymbol{d}_{Qy} + \omega_z \boldsymbol{d}_{Qz} + v_{Qx}\boldsymbol{d}_x + v_{Qy}\boldsymbol{d}_y + v_{Qz}\boldsymbol{d}_z \\ &\quad= \omega_x(\boldsymbol{d}_{Ox} - l\boldsymbol{d}_z) + \omega_y \boldsymbol{d}_{Oy} + \omega_z(\boldsymbol{d}_{Oz} + l\boldsymbol{d}_x) + (v_{Ox} - l\omega_z)\boldsymbol{d}_x + v_{Oy}\boldsymbol{d}_y + (v_{Oz} + l\omega_x)\boldsymbol{d}_z \\ &\quad= \omega_x \boldsymbol{d}_{Ox} + \omega_y \boldsymbol{d}_{Oy} + \omega_z \boldsymbol{d}_{Oz} + v_{Ox}\boldsymbol{d}_x + v_{Oy}\boldsymbol{d}_y + v_{Oz}\boldsymbol{d}_z\,. \end{aligned}$$

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Question B1

(a) $\boldsymbol{s}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{s}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

(b) $\boldsymbol{v}_1 = \boldsymbol{s}_1 \, \dot{q}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \boldsymbol{v}_2 = \boldsymbol{v}_1 + \boldsymbol{s}_2 \, \dot{q}_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \\ 0 \\ -\dot{q}_2 \\ 0 \end{bmatrix}$

(c) $\boldsymbol{J} = [\boldsymbol{s}_1 \; \boldsymbol{s}_2] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$

(d) $\boldsymbol{v}_P = \boldsymbol{v}_O - \overrightarrow{OP} \times \boldsymbol{\omega} = \begin{bmatrix} 0 \\ -\dot{q}_2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 + \dot{q}_2 \end{bmatrix} = \begin{bmatrix} -\dot{q}_1 - \dot{q}_2 \\ \dot{q}_1 \\ 0 \end{bmatrix}$

(where O is the origin, and $\boldsymbol{\omega}$ and $\boldsymbol{v}_O$ refer to $\hat{\boldsymbol{v}}_2$)

Question B2

(a) Let $\hat{\boldsymbol{f}}$ be the spatial force equivalent to a 3D force of $\boldsymbol{f}$ acting on a line passing through P. The Plücker coordinates of $\hat{\boldsymbol{f}}$ are therefore

$$\hat{\boldsymbol{f}} = \begin{bmatrix} \overrightarrow{OP} \times \boldsymbol{f} \\ \boldsymbol{f} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

Let $\hat{\boldsymbol{f}}_1$ and $\hat{\boldsymbol{f}}_2$ be the forces transmitted from the base to B_1 through joint 1, and from B_1 to B_2 through joint 2, respectively. For static equilibrium, the net force on each body must be zero. The net force on B_1 is $\hat{\boldsymbol{f}}_1 - \hat{\boldsymbol{f}}_2$, and the net force on B_2 is $\hat{\boldsymbol{f}}_2 + \hat{\boldsymbol{f}}$; so the condition for static equilibrium is

$$\hat{\boldsymbol{f}}_1 = \hat{\boldsymbol{f}}_2 = -\hat{\boldsymbol{f}} = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

(b) $\tau_1 = \boldsymbol{s}_1^{\mathrm{T}} \hat{\boldsymbol{f}}_1 = -1$ and $\tau_2 = \boldsymbol{s}_2^{\mathrm{T}} \hat{\boldsymbol{f}}_2 = -1$.

Question C1

$$\boldsymbol{a}_1 = \boldsymbol{s}_1\,\ddot{q}_1 + \dot{\boldsymbol{s}}_1\,\dot{q}_1 = \boldsymbol{s}_1\,\ddot{q}_1 = \begin{bmatrix}0\\0\\\ddot{q}_1\\0\\0\\0\end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{a}_2 &= \boldsymbol{a}_1 + \boldsymbol{s}_2\,\ddot{q}_2 + \dot{\boldsymbol{s}}_2\,\dot{q}_2\\
&= \boldsymbol{a}_1 + \boldsymbol{s}_2\,\ddot{q}_2 + \boldsymbol{v}_1 \times \boldsymbol{s}_2\,\dot{q}_2\\
&= \begin{bmatrix}0\\0\\\ddot{q}_1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\ddot{q}_2\\0\\-\ddot{q}_2\\0\end{bmatrix} + \begin{bmatrix}0\\0\\\dot{q}_1\\0\\0\\0\end{bmatrix} \times \begin{bmatrix}0\\0\\\dot{q}_2\\0\\-\dot{q}_2\\0\end{bmatrix} = \begin{bmatrix}0\\0\\\ddot{q}_1+\ddot{q}_2\\0\\-\ddot{q}_2\\0\end{bmatrix} + \begin{bmatrix}0\\0\\0\\\dot{q}_1\,\dot{q}_2\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\\ddot{q}_1+\ddot{q}_2\\\dot{q}_1\,\dot{q}_2\\-\ddot{q}_2\\0\end{bmatrix}
\end{aligned}$$

Question C2

Let C denote the position of a point on the central axis of the cylinder. The coordinates of C are then $(0, y_0 + v\,t, r)$, where y_0 is the y coordinate of C at $t = 0$. The angular velocity of the cylinder is $\boldsymbol{\omega} = [\,-v/r\ 0\ 0\,]^{\mathrm{T}}$, and the linear velocity at C is $\boldsymbol{v}_C = [\,0\ v\ 0\,]^{\mathrm{T}}$. The linear velocity at O is therefore

$$\boldsymbol{v}_O = \boldsymbol{v}_C + \overrightarrow{OC} \times \boldsymbol{\omega} = \begin{bmatrix}0\\v\\0\end{bmatrix} + \begin{bmatrix}0\\y_0+v\,t\\r\end{bmatrix} \times \begin{bmatrix}-v/r\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\(y_0+v\,t)\,v/r\end{bmatrix}.$$

Let $\hat{\boldsymbol{a}}_O$ be the coordinate vector expressing the spatial acceleration of the cylinder at O. As O is a fixed point in space, $\hat{\boldsymbol{a}}_O$ is just the componentwise derivative of the spatial velocity, $\hat{\boldsymbol{v}}_O$:

$$\hat{\boldsymbol{a}}_O = \frac{\mathrm{d}}{\mathrm{d}t}\,\hat{\boldsymbol{v}}_O = \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}-v/r\\0\\0\\0\\0\\(y_0+v\,t)\,v/r\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\\0\\v^2/r\end{bmatrix}.$$

Note: if we wish to perform this calculation at the moving point C, instead of the fixed point O, then we must calculate $\hat{\boldsymbol{a}}_C$ using the formula for differentiation in a moving Plücker coordinate system.

Question D1

Substitute $\boldsymbol{a} = \boldsymbol{S}\dot{\boldsymbol{\alpha}} + \dot{\boldsymbol{S}}\boldsymbol{\alpha}$ into the equation of motion:

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}(\boldsymbol{S}\dot{\boldsymbol{\alpha}} + \dot{\boldsymbol{S}}\boldsymbol{\alpha}) + \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v}\,.$$

Find $\dot{\boldsymbol{\alpha}}$:

$$\boldsymbol{I}\boldsymbol{S}\dot{\boldsymbol{\alpha}} = \boldsymbol{f} + \boldsymbol{f}_c - \boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} - \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v}$$

$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S}\dot{\boldsymbol{\alpha}} = \boldsymbol{S}^{\mathrm{T}}(\boldsymbol{f} - \boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} - \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v})$$

$$\dot{\boldsymbol{\alpha}} = (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{f} - \boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} - \boldsymbol{v}\times^{*}\boldsymbol{I}\boldsymbol{v})\,.$$

Substitute this expression for $\dot{\boldsymbol{\alpha}}$ back into $\boldsymbol{a} = \boldsymbol{S}\dot{\boldsymbol{\alpha}} + \dot{\boldsymbol{S}}\boldsymbol{\alpha}$:

$$\boldsymbol{a} = \boldsymbol{S}(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{f} - \boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} - \boldsymbol{v}\times^{*}\boldsymbol{I}\boldsymbol{v}) + \dot{\boldsymbol{S}}\boldsymbol{\alpha}\,.$$

This equation can be expressed in the form

$$\boldsymbol{a} = \boldsymbol{\Phi}\boldsymbol{f} + \boldsymbol{b}$$

where $\boldsymbol{\Phi}$ and $\boldsymbol{b}$ are the apparent inverse inertia and bias acceleration of the constrained body, respectively, and are given by

$$\boldsymbol{\Phi} = \boldsymbol{S}(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}$$

and

$$\boldsymbol{b} = \dot{\boldsymbol{S}}\boldsymbol{\alpha} - \boldsymbol{\Phi}(\boldsymbol{I}\dot{\boldsymbol{S}}\boldsymbol{\alpha} + \boldsymbol{v}\times^{*}\boldsymbol{I}\boldsymbol{v})\,.$$