A Short Course on

Spatial Vector Algebra

The Easy Way to do Rigid Body Dynamics

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Spatial vector algebra is a concise vector notation for describing rigid−body velocity, acceleration, inertia, etc., using 6D vectors and tensors.

- fewer quantities
- fewer equations
- less effort
- fewer mistakes

Mathematical Structure

spatial vectors inhabit *two* vector spaces:

M^6 — motion vectors
F^6 — force vectors

with a scalar product defined *between* them

$$\boldsymbol{m} \cdot \boldsymbol{f} = \mathit{work}$$

$$\text{“}\cdot\text{”} : \mathsf{M}^6 \times \mathsf{F}^6 \mapsto \mathsf{R}$$

Bases

A coordinate vector $\underline{\boldsymbol{m}} = [m_1, \ldots, m_6]^{\mathrm{T}}$ represents a motion vector $\boldsymbol{m}$ in a basis $\{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_6\}$ on M^6 if

$$\boldsymbol{m} = \sum_{i=1}^{6} m_i \boldsymbol{d}_i$$

Likewise, a coordinate vector $\underline{\boldsymbol{f}} = [f_1, \ldots, f_6]^{\mathrm{T}}$ represents a force vector $\boldsymbol{f}$ in a basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_6\}$ on F^6 if

$$\boldsymbol{f} = \sum_{i=1}^{6} f_i \boldsymbol{e}_i$$

Bases

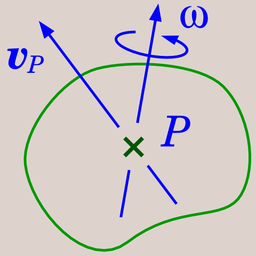
If $\{\boldsymbol{d}_1, \ldots, \boldsymbol{d}_6\}$ is an arbitrary basis on M^6 then there exists a unique *reciprocal basis* $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_6\}$ on F^6 satisfying

$$\boldsymbol{d}_i \cdot \boldsymbol{e}_j = \begin{cases} 0 : & i \neq j \\ 1 : & i = j \end{cases}$$

With these bases, the scalar product of two coordinate vectors is

$$\boldsymbol{m} \cdot \boldsymbol{f} = \underline{\boldsymbol{m}}^{\mathrm{T}} \underline{\boldsymbol{f}}$$

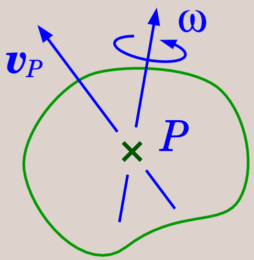
Velocity



The velocity of a rigid body can be described by

1. choosing a point, P, in the body
2. specifying the linear velocity, $\boldsymbol{v}_P$, of that point, and
3. specifying the angular velocity, ω, of the body as a whole

Velocity



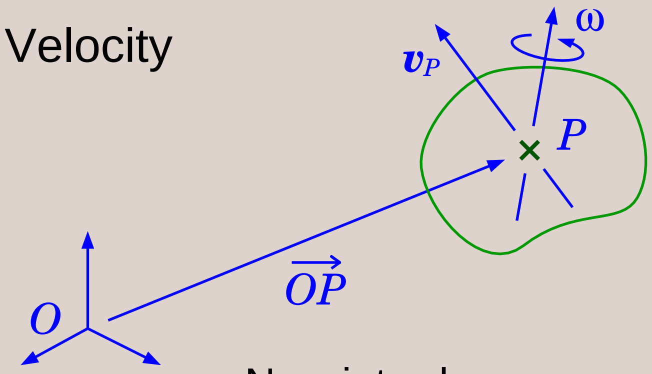
The body is then deemed to be

translating with a linear velocity $\boldsymbol{v}_P$

while simultaneously

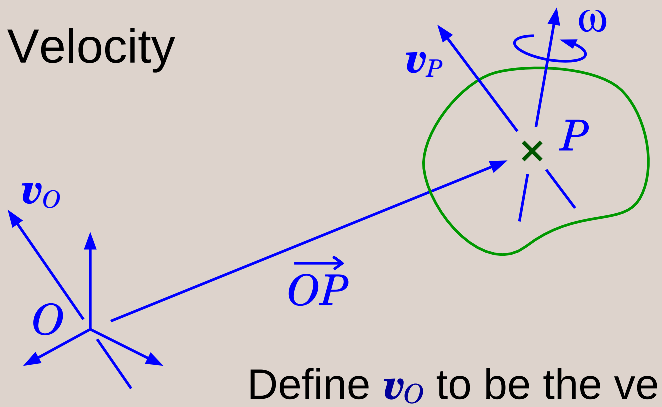
rotating with an angular velocity ω about an axis passing through P

Velocity



Now introduce a coordinate frame with an origin at any fixed point O

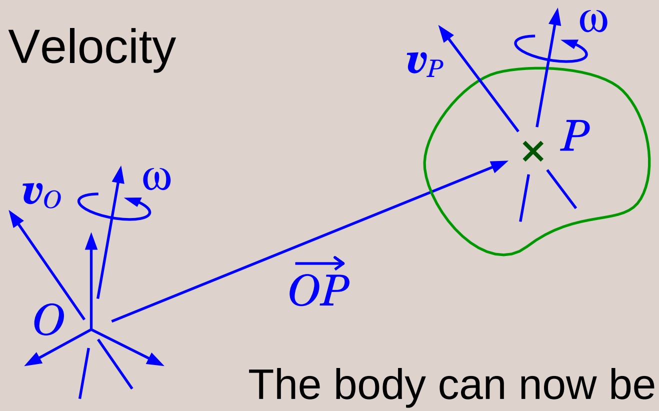
Velocity



Define $\boldsymbol{v}_O$ to be the velocity of the body−fixed point that coincides with O at the current instant

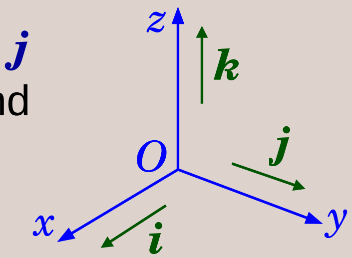
$$\boldsymbol{v}_O = \boldsymbol{v}_P + \overrightarrow{OP} \times \omega$$

Velocity



The body can now be regarded as translating with a velocity of $\boldsymbol{v}_O$ while simultaneously rotating with an angular velocity of ω about an axis passing through O

Introduce the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ pointing in the x, y and z directions.



ω and $\boldsymbol{v}_O$ can now be expressed in terms of their Cartesian coordinates:

$$\underline{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \quad \underline{v}_O = \begin{bmatrix} v_{Ox} \\ v_{Oy} \\ v_{Oz} \end{bmatrix}$$

coordinate vectors

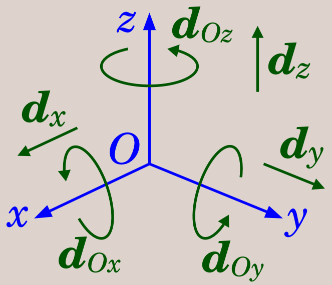
$$\omega = \omega_x \boldsymbol{i} + \omega_y \boldsymbol{j} + \omega_z \boldsymbol{k}$$
$$\boldsymbol{v}_O = v_{Ox} \boldsymbol{i} + v_{Oy} \boldsymbol{j} + v_{Oz} \boldsymbol{k}$$

what they represent

The motion of the body can now be expressed as the sum of six elementary motions:

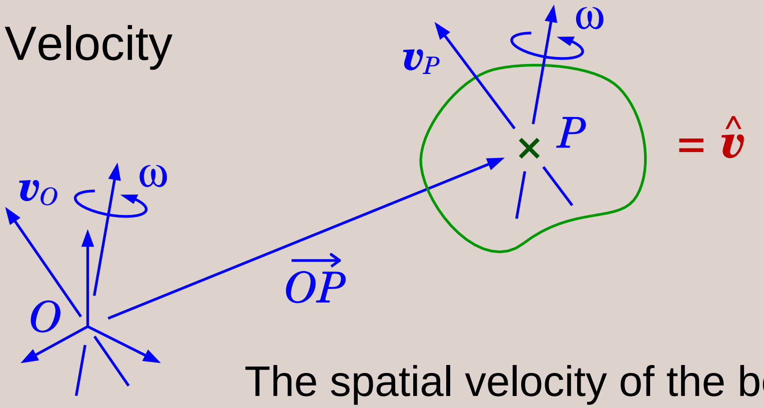
a linear velocity of v_{Ox} in the x direction
\+ a linear velocity of v_{Oy} in the y direction
\+ a linear velocity of v_{Oz} in the z direction
\+ an angular velocity of ω_x about the line Ox
\+ an angular velocity of ω_y about the line Oy
\+ an angular velocity of ω_z about the line Oz

Define the following *Plücker basis* on M^6:



$\boldsymbol{d}_{Ox}$ unit angular motion about the line Ox
$\boldsymbol{d}_{Oy}$ unit angular motion about the line Oy
$\boldsymbol{d}_{Oz}$ unit angular motion about the line Oz
$\boldsymbol{d}_x$ unit linear motion in the x direction
$\boldsymbol{d}_y$ unit linear motion in the y direction
$\boldsymbol{d}_z$ unit linear motion in the z direction

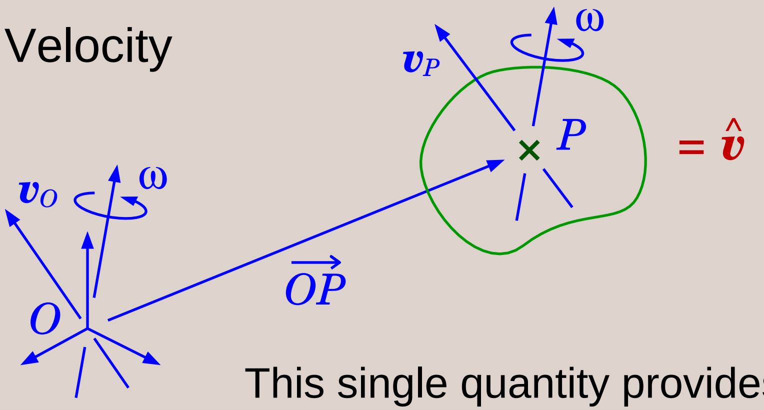
Velocity



The spatial velocity of the body can now be expressed as

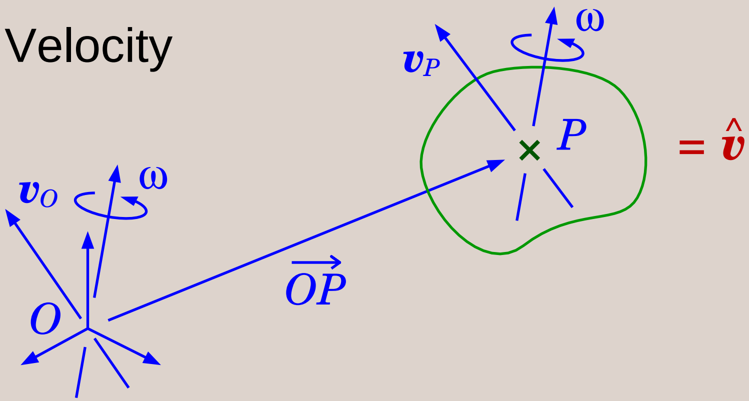
$$\hat{\boldsymbol{v}} = \omega_x \boldsymbol{d}_{Ox} + \omega_y \boldsymbol{d}_{Oy} + \omega_z \boldsymbol{d}_{Oz} + \\ + v_{Ox} \boldsymbol{d}_x + v_{Oy} \boldsymbol{d}_y + v_{Oz} \boldsymbol{d}_z$$

Velocity



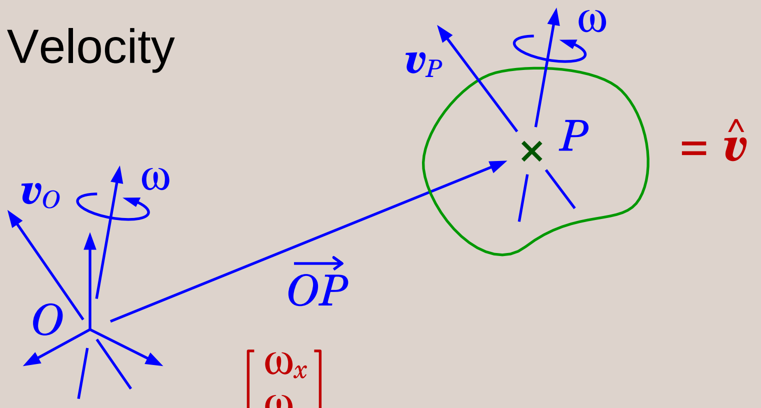
This single quantity provides a *complete description* of the velocity of a rigid body, and it is *invariant* with respect to the location of the coordinate frame

Velocity



The six scalars ω_x, ω_y,…, v_{Oz} are the *Plücker coordinates* of $\hat{\boldsymbol{v}}$ in the coordinate system defined by the frame $Oxyz$

Velocity



$= \hat{\boldsymbol{v}}$

$$\hat{\underline{v}}_O = \begin{bmatrix} \underline{\omega} \\ \underline{v}_O \end{bmatrix} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \\ v_{Ox} \\ v_{Oy} \\ v_{Oz} \end{bmatrix}$$

coordinate vector

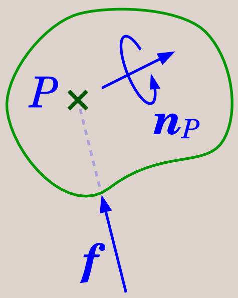
$$\hat{\boldsymbol{v}} = \omega_x \boldsymbol{d}_{Ox} + \omega_y \boldsymbol{d}_{Oy} + \omega_z \boldsymbol{d}_{Oz} + v_{Ox} \boldsymbol{d}_x + v_{Oy} \boldsymbol{d}_y + v_{Oz} \boldsymbol{d}_z$$

what it represents

Now try question set A

Force

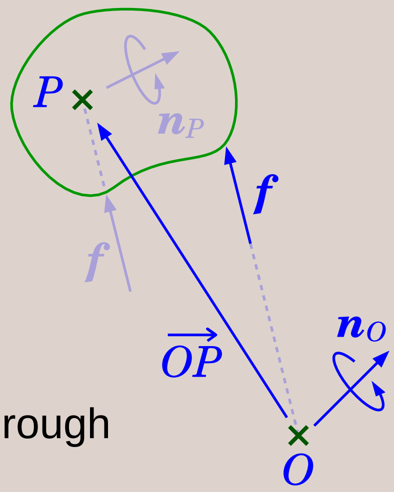
A general force acting on a rigid body can be expressed as the sum of



- a linear force $\boldsymbol{f}$ acting along a line passing through any chosen point P, and
- a couple, $\boldsymbol{n}_P$

Force

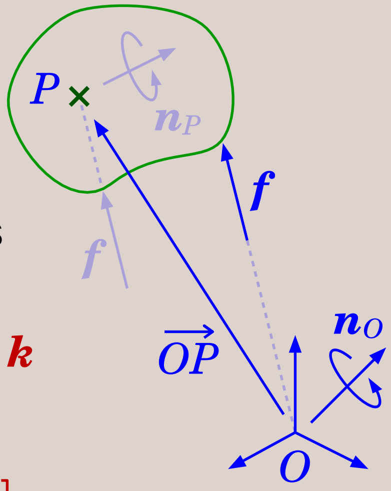
If we choose a different point, O, then the force can be expressed as the sum of



- a linear force $\boldsymbol{f}$ acting along a line passing through the new point O, and
- a couple $\boldsymbol{n}_O$, where $\boldsymbol{n}_O = \boldsymbol{n}_P + \overrightarrow{OP} \times \boldsymbol{f}$

Force

Now place a coordinate frame at O and introduce unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, as before, so that



$$\boldsymbol{n}_O = n_{Ox}\,\boldsymbol{i} + n_{Oy}\,\boldsymbol{j} + n_{Oz}\,\boldsymbol{k}$$

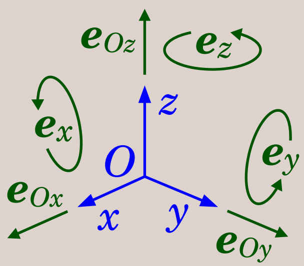
$$\boldsymbol{f} = f_x\,\boldsymbol{i} + f_y\,\boldsymbol{j} + f_z\,\boldsymbol{k}$$

$$\underline{\boldsymbol{n}}_O = \begin{bmatrix} n_{Ox} \\ n_{Oy} \\ n_{Oz} \end{bmatrix} \qquad \underline{\boldsymbol{f}} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

The total force acting on the body can now be expressed as the sum of six elementary forces:

a moment of n_{Ox} in the x direction
\+ a moment of n_{Oy} in the y direction
\+ a moment of n_{Oz} in the z direction
\+ a linear force of f_x acting along the line Ox
\+ a linear force of f_y acting along the line Oy
\+ a linear force of f_z acting along the line Oz

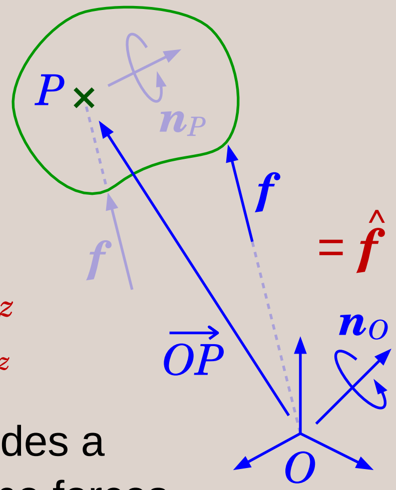
Define the following *Plücker basis* on F^6:



$\boldsymbol{e}_x$	unit couple in the x direction
$\boldsymbol{e}_y$	unit couple in the y direction
$\boldsymbol{e}_z$	unit couple in the z direction
$\boldsymbol{e}_{Ox}$	unit linear force along the line Ox
$\boldsymbol{e}_{Oy}$	unit linear force along the line Oy
$\boldsymbol{e}_{Oz}$	unit linear force along the line Oz

Force

The spatial force acting on the body can now be expressed as

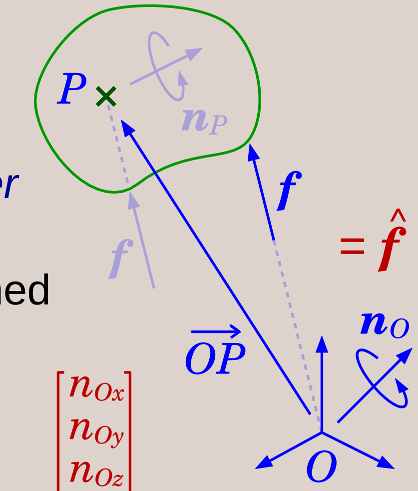


$$\hat{\boldsymbol{f}} = n_{Ox}\,\boldsymbol{e}_x + n_{Oy}\,\boldsymbol{e}_y + n_{Oz}\,\boldsymbol{e}_z + f_x\,\boldsymbol{e}_{Ox} + f_y\,\boldsymbol{e}_{Oy} + f_z\,\boldsymbol{e}_{Oz}$$

This single quantity provides a *complete description* of the forces acting on the body, and it is *invariant* with respect to the location of the coordinate frame

Force

The six scalars n_{Ox}, $n_{Oy}, \ldots, f_z$ are the *Plücker coordinates* of $\hat{\boldsymbol{f}}$ in the coordinate system defined by the frame $Oxyz$



coordinate vector: $\underline{\hat{f}}_O = \begin{bmatrix} \underline{\boldsymbol{n}}_O \\ \underline{\boldsymbol{f}} \end{bmatrix} = \begin{bmatrix} n_{Ox} \\ n_{Oy} \\ n_{Oz} \\ f_x \\ f_y \\ f_z \end{bmatrix}$

Plücker Coordinates

- Plücker coordinates are the standard coordinate system for spatial vectors
- a Plücker coordinate system is defined by the *position and orientation* of a *single* Cartesian frame
- a Plücker coordinate system has a total of *twelve* basis vectors, and covers both vector spaces (M^6 and F^6)

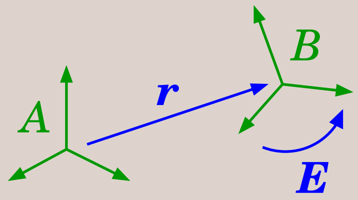
Plücker Coordinates

- the Plücker basis $\boldsymbol{e}_x, \boldsymbol{e}_y, \ldots, \boldsymbol{e}_{Oz}$ on F^6 is reciprocal to $\boldsymbol{d}_{Ox}, \boldsymbol{d}_{Oy}, \ldots, \boldsymbol{d}_z$ on M^6
- so the scalar product between a motion vector and a force vector can be expressed in Plücker coordinates as

$$\hat{\boldsymbol{v}} \cdot \hat{\boldsymbol{f}} = \underline{\hat{v}}_O^{\mathrm{T}} \, \underline{\hat{f}}_O$$

which is invariant with respect to the location of the coordinate frame

Coordinate Transforms



transform from A to B for motion vectors:

$${}^{B}\boldsymbol{X}_A = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} \\ \tilde{\boldsymbol{r}}^{\mathrm{T}} & \boldsymbol{1} \end{bmatrix} \quad \text{where} \quad \tilde{\boldsymbol{r}} = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$$

corresponding transform for force vectors: ${}^{B}\boldsymbol{X}_A^{*} = ({}^{B}\boldsymbol{X}_A)^{-\mathrm{T}}$

Spatial Vector Algebra

Basic Operations with Spatial Vectors

- Relative velocity

  If bodies A and B have velocities of $\boldsymbol{v}_A$ and $\boldsymbol{v}_B$, then the relative velocity of B with respect to A is

  $$\boldsymbol{v}_{\text{rel}} = \boldsymbol{v}_B - \boldsymbol{v}_A$$

- Rigid Connection

  If two bodies are rigidly connected then their velocities are the same

Spatial Vector Algebra

- Summation of Forces

  If forces $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ both act on the same body, then they are equivalent to a single force $\boldsymbol{f}_{\text{tot}}$ given by

  $$\boldsymbol{f}_{\text{tot}} = \boldsymbol{f}_1 + \boldsymbol{f}_2$$

- Action and Reaction

  If body A exerts a force $\boldsymbol{f}$ on body B, then body B exerts a force $-\boldsymbol{f}$ on body A (Newton's 3rd law)

Spatial Vector Algebra

- Scalar Product

  If a force $\boldsymbol{f}$ acts on a body with velocity $\boldsymbol{v}$, then the power delivered by that force is

  $$power = \boldsymbol{v} \cdot \boldsymbol{f}$$

- Scalar Multiples

  A velocity of $\alpha\boldsymbol{v}$ causes the same movement in 1 second as a velocity of $\boldsymbol{v}$ in α seconds. A force of $\beta\boldsymbol{f}$ delivers β times as much power as a force of $\boldsymbol{f}$

Spatial Vector Algebra

Now try question set B

Spatial Cross Products

There are *two* cross product operations: one for motion vectors and one for forces

$$\hat{v}_O \times \hat{m}_O = \begin{bmatrix} \omega \\ v_O \end{bmatrix} \times \begin{bmatrix} m \\ m_O \end{bmatrix} = \begin{bmatrix} \omega \times m \\ \omega \times m_O + v_O \times m \end{bmatrix}$$

$$\hat{v}_O \times^* \hat{f}_O = \begin{bmatrix} \omega \\ v_O \end{bmatrix} \times^* \begin{bmatrix} n_O \\ f \end{bmatrix} = \begin{bmatrix} \omega \times n_O + v_O \times f \\ \omega \times f \end{bmatrix}$$

where $\hat{v}_O$ and $\hat{m}_O$ are motion vectors, and $\hat{f}_O$ is a force.

Differentiation

- The derivative of a spatial vector is itself a spatial vector
- in general, $\frac{d}{dt} s = \lim_{\delta t \to 0} \frac{s(t+\delta t) - s(t)}{\delta t}$
- The derivative of a spatial vector that is fixed in a body moving with velocity v is

$$\frac{d}{dt} s = \begin{cases} v \times s & \text{if } s \in M^6 \\ v \times^* s & \text{if } s \in F^6 \end{cases}$$

Differentiation in Moving Coordinates

$$\left[\frac{d}{dt} s\right]_O = \frac{d}{dt} s_O + v_O \times s_O$$

- $\times$: or $\times^*$ if $s \in F^6$
- v_O: velocity of coordinate frame
- $\frac{d}{dt} s_O$: componentwise derivative of coordinate vector s_O
- $\left[\frac{d}{dt} s\right]_O$: coordinate vector representing ds/dt

Acceleration

. . . is the rate of change of velocity:

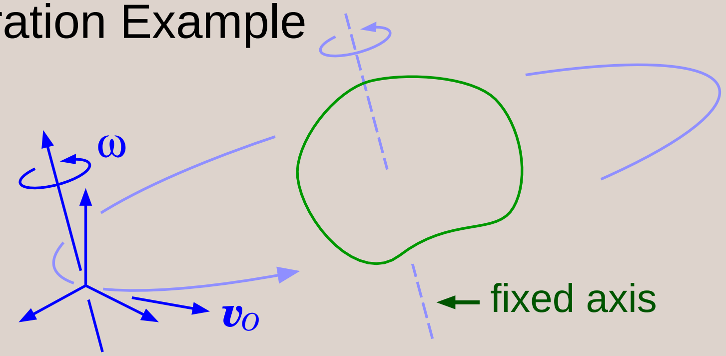
$$\hat{a} = \frac{d}{dt} \hat{v} = \begin{bmatrix} \dot{\omega} \\ \dot{v}_O \end{bmatrix}$$

$\dot{v}_O$: but this is *not* the linear acceleration of any point in the body!

Acceleration

- O is a fixed point in space,
- and $\boldsymbol{v}_O(t)$ is the velocity of the body−fixed point that coincides with O at time t,
- so $\boldsymbol{v}_O$ is the velocity at which body−fixed points are streaming through O.
- $\dot{\boldsymbol{v}}_O$ is therefore the rate of change of stream velocity

Acceleration Example



If a body rotates with constant angular velocity about a fixed axis, then its spatial velocity is constant and its spatial acceleration is zero; but each body−fixed point is following a circular path, and is therefore accelerating.

Acceleration Formula

Let $\boldsymbol{r}$ be the 3D vector giving the position of the body−fixed point that coincides with O at the current instant, measured relative to any fixed point in space

we then have $\hat{\boldsymbol{v}} = \begin{bmatrix} \omega \\ \boldsymbol{v}_O \end{bmatrix} = \begin{bmatrix} \omega \\ \dot{\boldsymbol{r}} \end{bmatrix}$

but $\hat{\boldsymbol{a}} = \begin{bmatrix} \dot{\omega} \\ \dot{\boldsymbol{v}}_O \end{bmatrix} = \begin{bmatrix} \dot{\omega} \\ \ddot{\boldsymbol{r}} - \omega \times \dot{\boldsymbol{r}} \end{bmatrix}$

Basic Properties of Acceleration

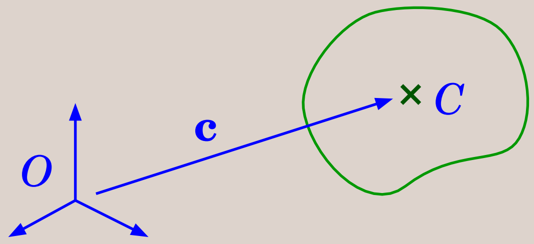
- Acceleration is the time−derivative of velocity
- Acceleration is a true vector, and has the same general algebraic properties as velocity
- Acceleration formulae are the derivatives of velocity formulae

If $\boldsymbol{v}_{tot} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ then $\boldsymbol{a}_{tot} = \boldsymbol{a}_1 + \boldsymbol{a}_2$

(Look, no Coriolis term!)

Now try question set C

Rigid Body Inertia



mass: m

CoM: C

inertia at CoM: $\boldsymbol{I}_C$

spatial inertia tensor: $$\hat{\boldsymbol{I}}_O = \begin{bmatrix} \boldsymbol{I}_O & m\tilde{\mathbf{c}} \\ m\tilde{\mathbf{c}}^{\mathrm{T}} & m\mathbf{1} \end{bmatrix}$$

where $\boldsymbol{I}_O = \boldsymbol{I}_C - m\,\tilde{\mathbf{c}}\,\tilde{\mathbf{c}}$

Basic Operations with Inertias

- Composition

  If two bodies with inertias $\boldsymbol{I}_A$ and $\boldsymbol{I}_B$ are joined together then the inertia of the composite body is

  $$\boldsymbol{I}_{\text{tot}} = \boldsymbol{I}_A + \boldsymbol{I}_B$$

- Coordinate transformation formula

  $$\boldsymbol{I}_B = {}^B\boldsymbol{X}_A^*\,\boldsymbol{I}_A\,{}^A\boldsymbol{X}_B = ({}^A\boldsymbol{X}_B)^{\mathrm{T}}\,\boldsymbol{I}_A\,{}^A\boldsymbol{X}_B$$

Equation of Motion

$$\boldsymbol{f} = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{I}\boldsymbol{v}) = \boldsymbol{I}\boldsymbol{a} + \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v}$$

$\boldsymbol{f}$ = net force acting on a rigid body

$\boldsymbol{I}$ = inertia of rigid body

$\boldsymbol{v}$ = velocity of rigid body

$\boldsymbol{I}\boldsymbol{v}$ = momentum of rigid body

$\boldsymbol{a}$ = acceleration of rigid body

Motion Constraints

If a rigid body's motion is constrained, then its velocity is an element of a subspace, $S \subset \mathsf{M}^6$, called the *motion subspace*

degree of (motion) freedom: $\dim(S)$

degree of constraint: $6 - \dim(S)$

S can vary with time

Motion Constraints

Motion constraints are caused by constraint forces, which have the following property:

A constraint force does no work against any motion allowed by the motion constraint

(D'Alembert's principle of virtual work, and Jourdain's principle of virtual power)

Motion Constraints

Constraint forces are therefore elements of a constraint−force subspace, $T \subset \mathsf{F}^6$, defined as follows:

$$T = \{\, \boldsymbol{f} \mid \boldsymbol{f} \cdot \boldsymbol{v} = 0 \;\forall\; \boldsymbol{v} \in S \,\}$$

This subspace has the property

$$\dim(T) = 6 - \dim(S)$$

Matrix Representation

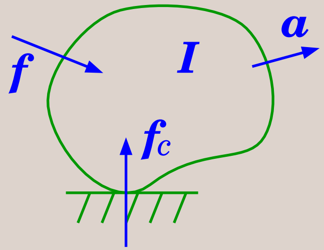
- The subspace S can be represented by any $6 \times \dim(S)$ matrix $\boldsymbol{S}$ satisfying $\mathrm{range}(\boldsymbol{S}) = S$
- Likewise, the subspace T can be represented by any $6 \times \dim(T)$ matrix $\boldsymbol{T}$ satisfying $\mathrm{range}(\boldsymbol{T}) = T$

Properties

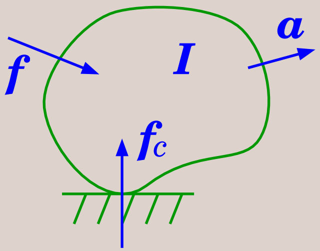
- any vectors $\boldsymbol{v} \in S$ and $\boldsymbol{f} \in T$ can be expressed as $\boldsymbol{v} = \boldsymbol{S}\alpha$ and $\boldsymbol{f} = \boldsymbol{T}\lambda$, where α and λ are $\dim(S) \times 1$ and $\dim(T) \times 1$ coordinate vectors
- $\boldsymbol{S}^{\mathrm{T}}\boldsymbol{T} = \mathbf{0}$, which implies . . .
- $\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f} = \mathbf{0}$ and $\boldsymbol{T}^{\mathrm{T}}\boldsymbol{v} = \mathbf{0}$ for all $\boldsymbol{f} \in T$ and $\boldsymbol{v} \in S$

Constrained Motion Analysis

An Example:



A force, $\boldsymbol{f}$, is applied to a body that is constrained to move in a subspace $S = \mathrm{range}(\boldsymbol{S})$ of M^6. The body has an inertia of $\boldsymbol{I}$, and it is initially at rest. What is its acceleration?



relevant equations:

$$\boldsymbol{v} = \boldsymbol{S}\alpha$$
$$\boldsymbol{a} = \boldsymbol{S}\dot{\alpha} + \dot{\boldsymbol{S}}\alpha$$
$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}_c = \mathbf{0}$$
$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{a} + \boldsymbol{v}\times^*\boldsymbol{I}\boldsymbol{v}$$

$\boldsymbol{v} = \mathbf{0}$ implies

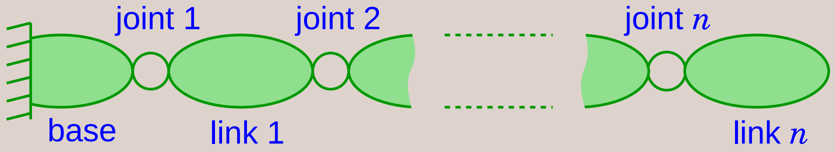
$$\alpha = \mathbf{0}$$
$$\boldsymbol{a} = \boldsymbol{S}\dot{\alpha}$$
$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{a}$$

solution:

$$\boldsymbol{f} + \boldsymbol{f}_c = \boldsymbol{I}\boldsymbol{S}\dot{\alpha}$$
$$\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f} = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S}\dot{\alpha}$$
$$\dot{\alpha} = (\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}$$
$$\boldsymbol{a} = \boldsymbol{S}(\boldsymbol{S}^{\mathrm{T}}\boldsymbol{I}\boldsymbol{S})^{-1}\boldsymbol{S}^{\mathrm{T}}\boldsymbol{f}$$

Now try question set D

Inverse Dynamics



$\dot{q}_i, \ddot{q}_i, \boldsymbol{s}_i$	joint velocity, acceleration & axis
$\boldsymbol{v}_i, \boldsymbol{a}_i$	link velocity and acceleration
$\boldsymbol{f}_i$	force transmitted from link $i-1$ to i
τ_i	joint force variable
$\boldsymbol{I}_i$	link inertia

- velocity of link i is the velocity of link $i-1$ plus the velocity across joint i

$$\boldsymbol{v}_i = \boldsymbol{v}_{i-1} + \boldsymbol{s}_i \dot{q}_i$$

- acceleration is the derivative of velocity

$$\boldsymbol{a}_i = \boldsymbol{a}_{i-1} + \dot{\boldsymbol{s}}_i \dot{q}_i + \boldsymbol{s}_i \ddot{q}_i$$

- equation of motion

$$\boldsymbol{f}_i - \boldsymbol{f}_{i+1} = \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{v}_i \times^* \boldsymbol{I}_i \boldsymbol{v}_i$$

- active joint force

$$\tau_i = \boldsymbol{s}_i^{\mathrm{T}} \boldsymbol{f}_i$$

The Recursive Newton−Euler Algorithm

(Calculate the joint torques τ_i that will produce the desired joint accelerations $\ddot{q}_i$.)

$$\boldsymbol{v}_i = \boldsymbol{v}_{i-1} + \boldsymbol{s}_i \dot{q}_i \qquad (\boldsymbol{v}_0 = \boldsymbol{0})$$

$$\boldsymbol{a}_i = \boldsymbol{a}_{i-1} + \dot{\boldsymbol{s}}_i \dot{q}_i + \boldsymbol{s}_i \ddot{q}_i \qquad (\boldsymbol{a}_0 = \boldsymbol{0})$$

$$\boldsymbol{f}_i = \boldsymbol{f}_{i+1} + \boldsymbol{I}_i \boldsymbol{a}_i + \boldsymbol{v}_i \times^* \boldsymbol{I}_i \boldsymbol{v}_i \qquad (\boldsymbol{f}_{n+1} = \boldsymbol{f}_{ee})$$

$$\tau_i = \boldsymbol{s}_i^{\mathrm{T}} \boldsymbol{f}_i$$