The Condition Number of the Joint Space Inertia Matrix

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The condition number of a matrix measures its closeness to singularity:

$$\kappa(\boldsymbol{A}) \longrightarrow \text{infinity} \quad \text{as} \quad \boldsymbol{A} \longrightarrow \text{singular}$$

If $\kappa(\boldsymbol{A})$ is large then $\boldsymbol{A}$ is said to be ill−conditioned.

If a physical system is described by an equation like

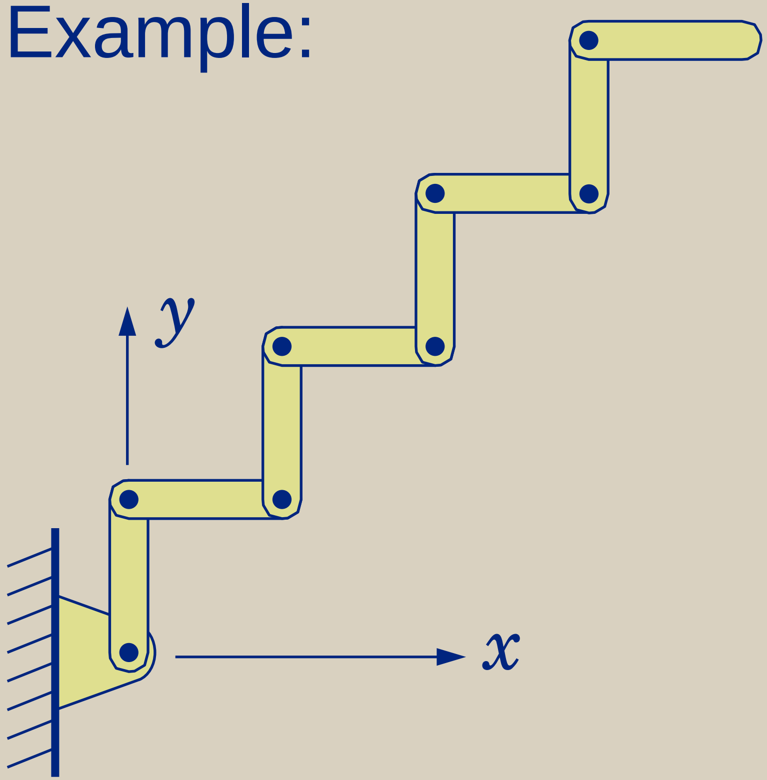
$$\boldsymbol{y} = \boldsymbol{A}\,\boldsymbol{x} + \boldsymbol{b}$$

and $\boldsymbol{A}$ is ill−conditioned, then it can be difficult to calculate $\boldsymbol{y}$ from $\boldsymbol{x}$, or $\boldsymbol{x}$ from $\boldsymbol{y}$, without loss of accuracy.

The joint−space inertia matrix (JSIM) of a kinematic tree is known to be a symmetric, positive−definite matrix.

- It is therefore nonsingular.
- But is it ill−conditioned? Yes!

Example:



Suppose we want to accelerate this planar 8R robot from rest with an acceleration of

$$\ddot{\boldsymbol{q}}_d = [1,1,1,1,1,1,1,1]^T$$

The equation of motion is $\tau = \boldsymbol{H}\,\ddot{\boldsymbol{q}} + \boldsymbol{C}$ where $\boldsymbol{H}$ is the JSIM, and $\boldsymbol{C} = \boldsymbol{0}$ (gravity and velocity terms are zero).

The exact force required to produce an acceleration of $\ddot{\boldsymbol{q}}_d$ is

$$\tau_d = \boldsymbol{H}\,\ddot{\boldsymbol{q}}_d = \begin{bmatrix} 302.0450\cdots \\ 250.2104\cdots \\ 200.5413\cdots \\ 151.0375\cdots \\ 105.6992\cdots \\ 64.5263\cdots \\ 31.5188\cdots \\ 8.6767\cdots \end{bmatrix}$$

But what if the actual joint force differs very slightly from the theoretically exact force?

Let τ_a be the exact force rounded to three significant figures. The acceleration caused by an applied force of τ_a is

$$\ddot{\boldsymbol{q}}_a = \boldsymbol{H}^{-1}\,\tau_a = \boldsymbol{H}^{-1}\begin{bmatrix} 302 \\ 250 \\ 201 \\ 151 \\ 106 \\ 64.5 \\ 31.5 \\ 8.68 \end{bmatrix} = \begin{bmatrix} 0.7917 \\ 1.0281 \\ 1.4904 \\ 0.6886 \\ 1.1026 \\ 1.0911 \\ 0.5626 \\ 1.2384 \end{bmatrix}$$

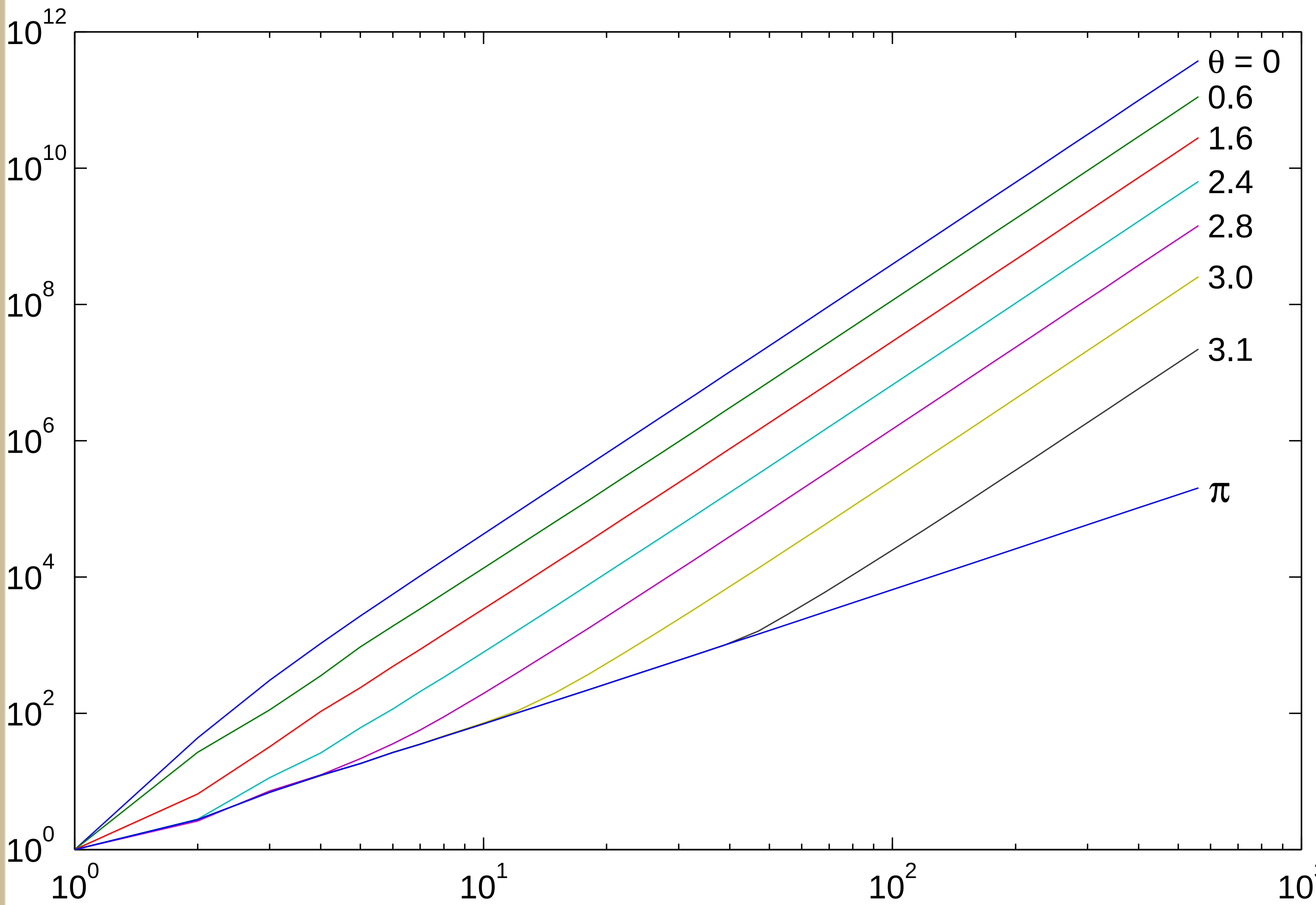
A force error of < 0.5% has caused an acceleration error of 50%

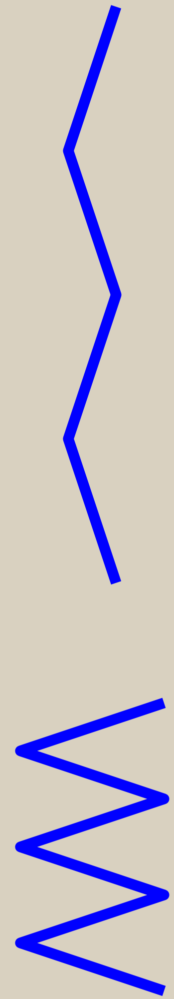
Measuring the Condition Number

The following graphs plot $\kappa(\boldsymbol{H})$ vs N (number of bodies) for robots with

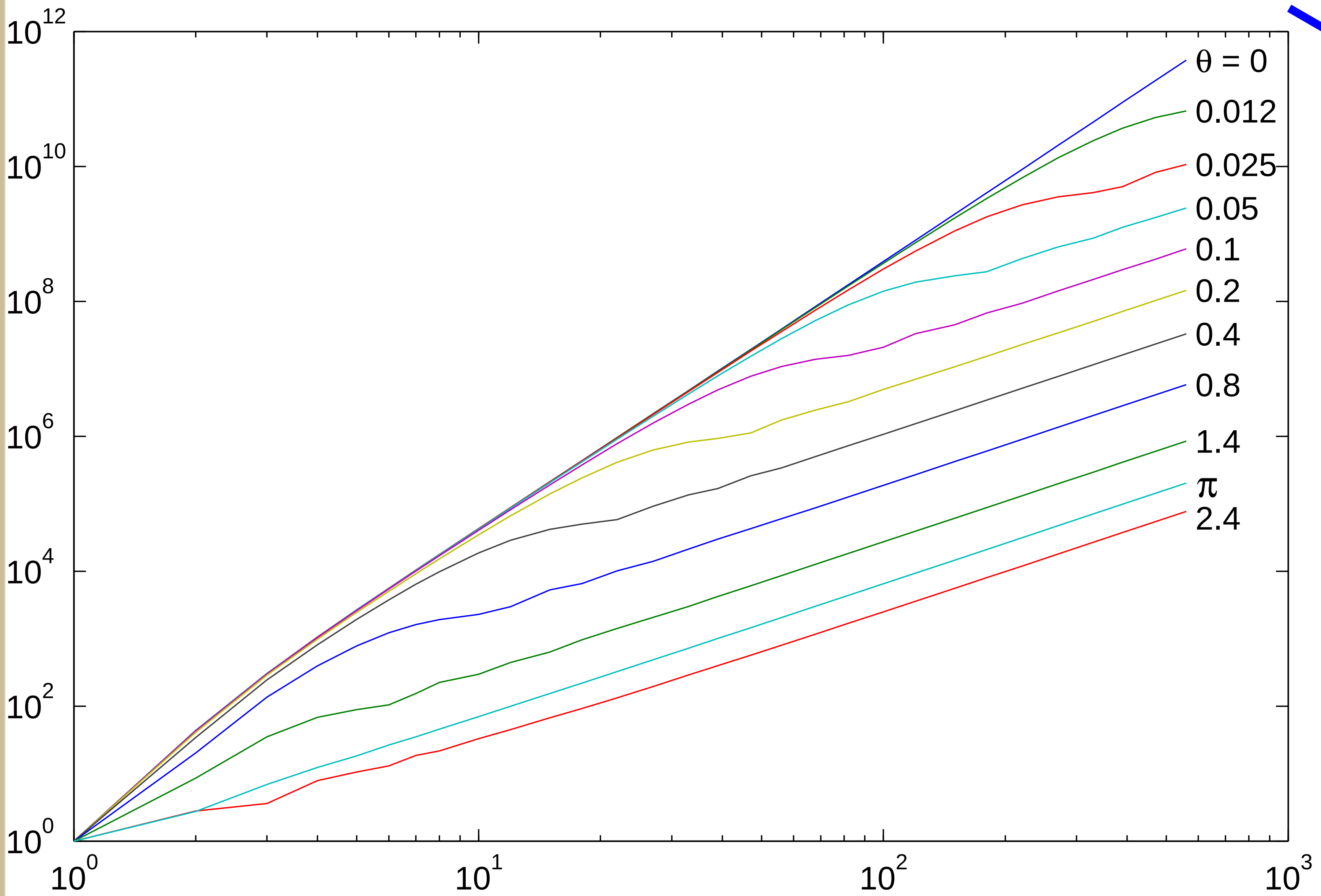
- revolute joints
- identical or tapering links
- in curled and zigzag configurations
- unbranched or branched connectivity
- fixed or floating bases

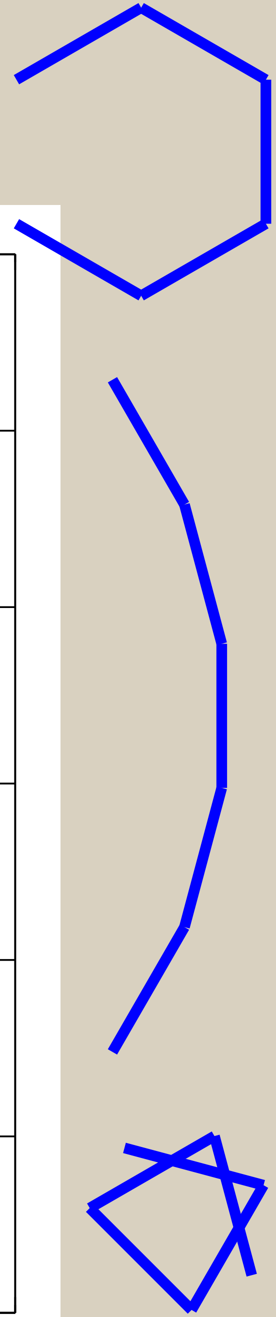
identical links, unbranched, zigzag



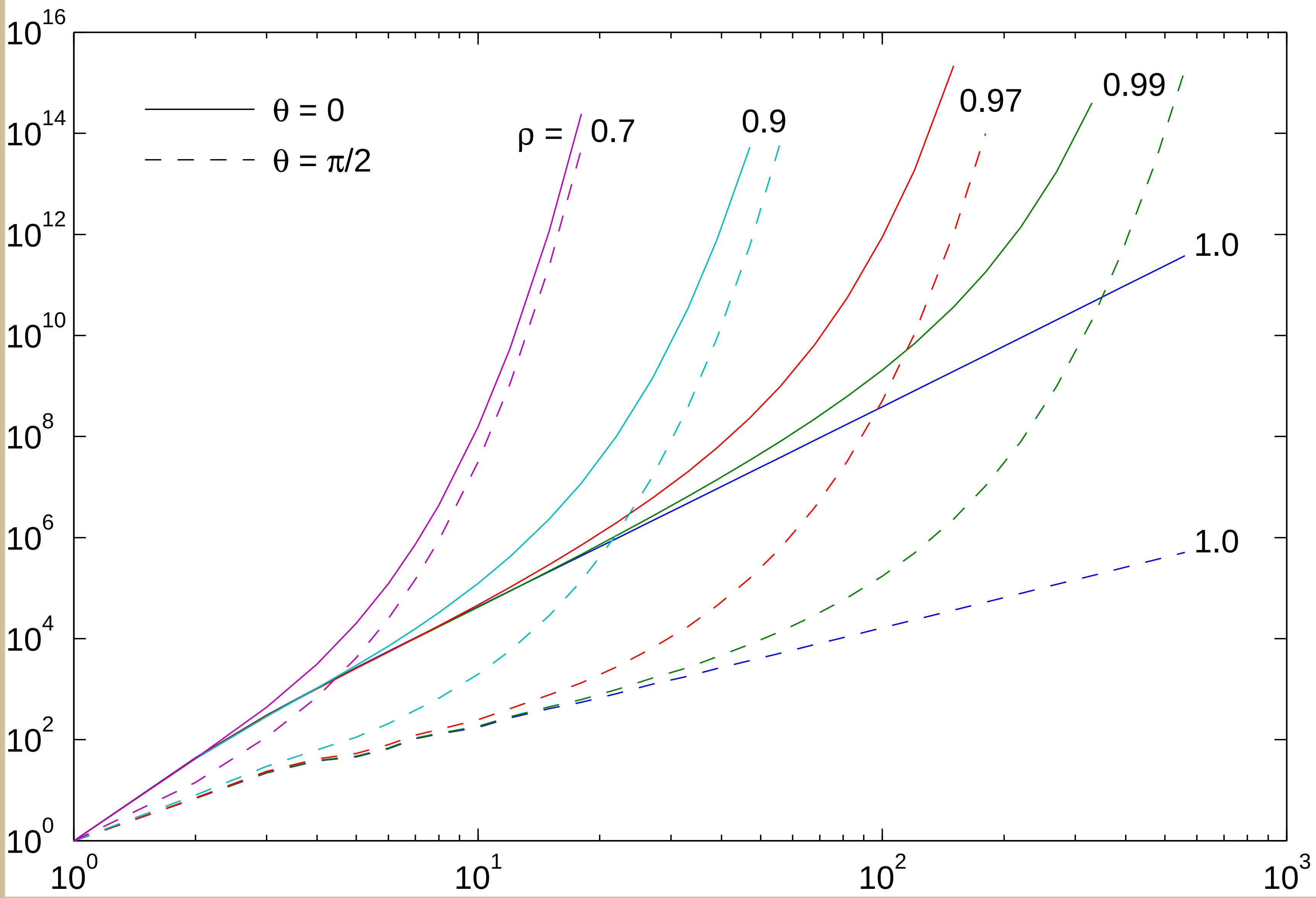


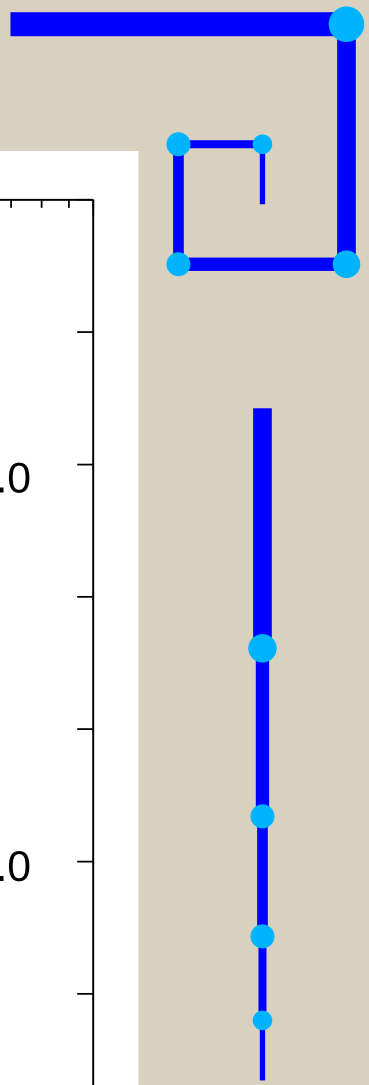
identical links, unbranched, curled



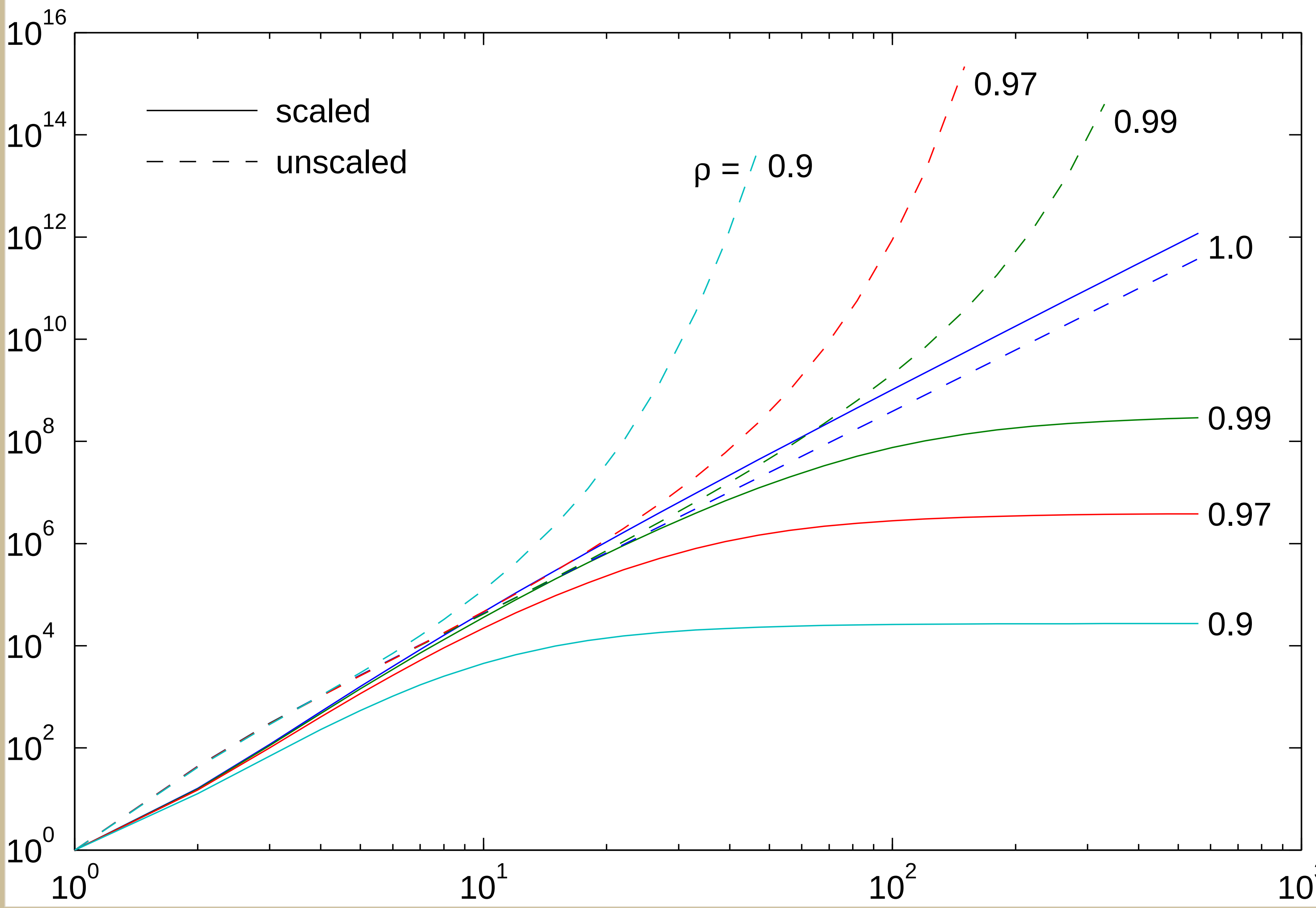


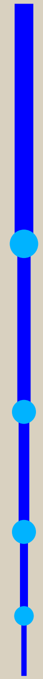
tapered links, unbranched, curled



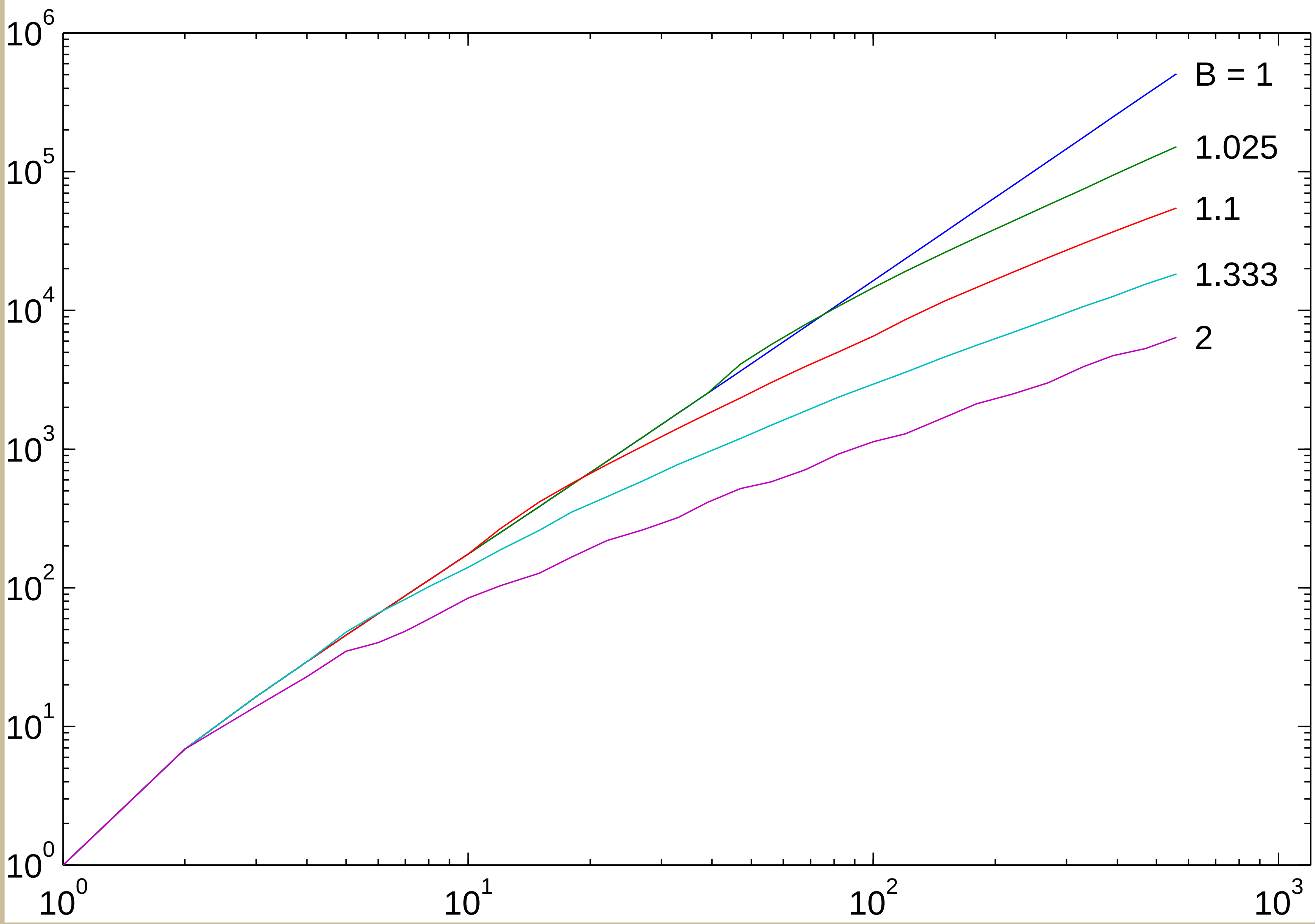


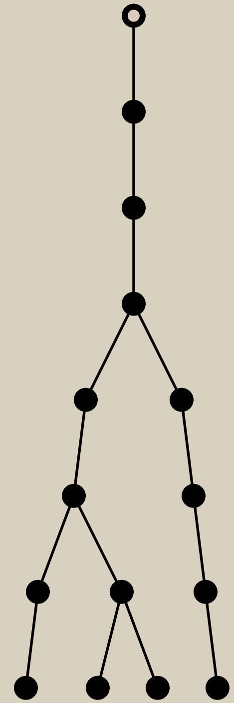
inertia−weighted metric





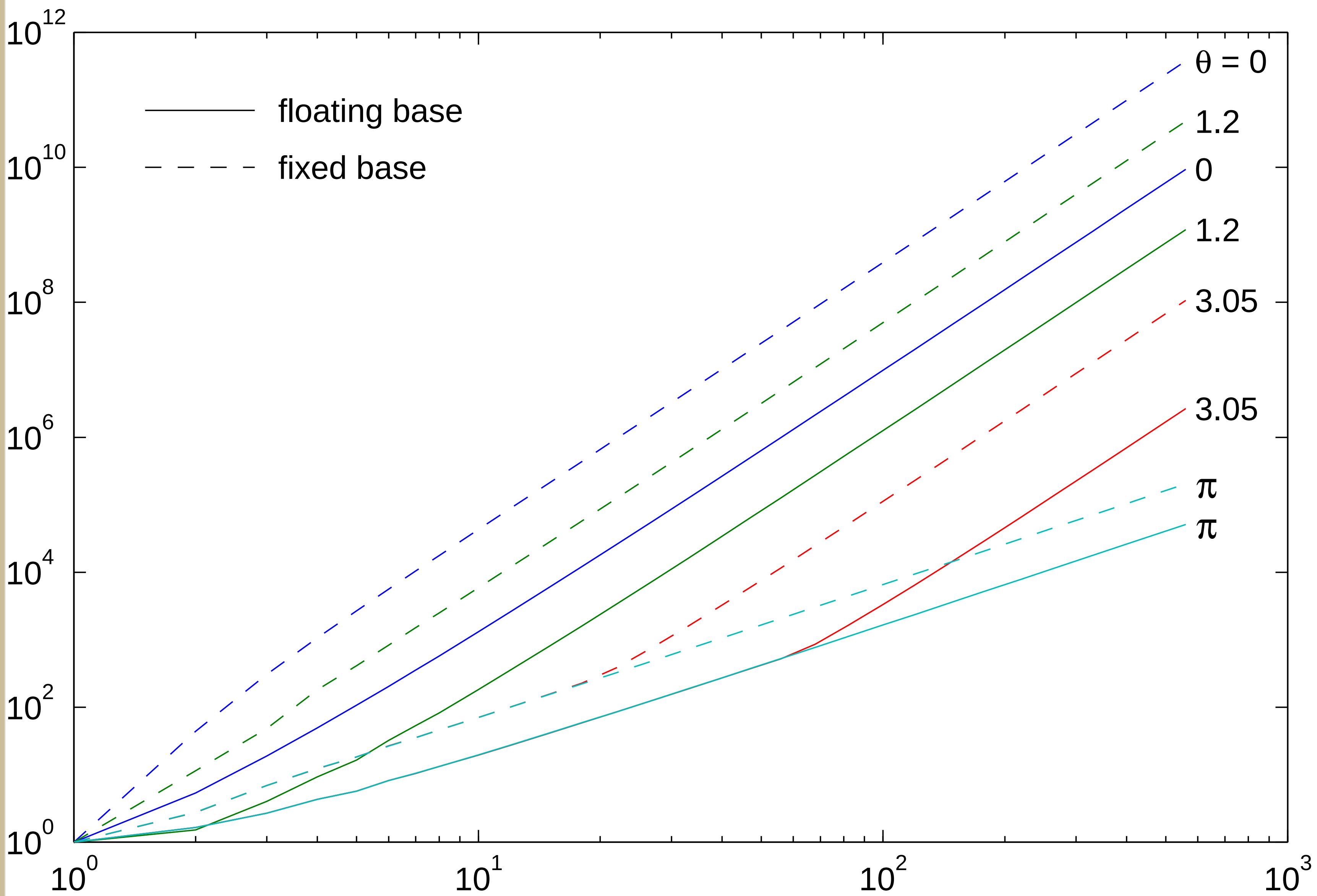
branches (spherically symmetric robots)

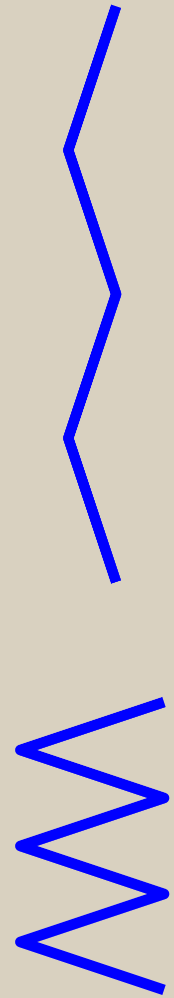




B=1.33

floating base, unbranched, identical links





Summary

- in general, the JSIM is very ill−conditioned, and it gets worse as the number of bodies increases
- worst case: $\kappa(\boldsymbol{H}) = 4N^4$
- tapering can increase or decrease ill−conditioning, depending on how you measure it
- branches reduce ill−conditioning
- a floating base can reduce ill−conditioning