Solving a Two-Body Dynamics Problem using 3-D Vectors

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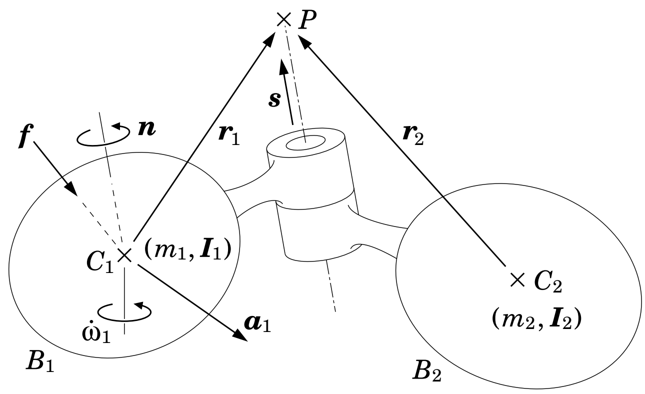
Problem Statement

We are given a rigid-body system consisting of two bodies, B_1 and B_2, connected by a revolute joint. The bodies have masses of m_1 and m_2, centres of mass located at the points C_1 and C_2, and rotational inertias of $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ about their respective centres of mass. Both bodies are initially at rest. The joint's axis of rotation passes through the point P in the direction given by $\boldsymbol{s}$. A system of forces acts on B_1 causing both bodies to accelerate. This system is equivalent to a single force $\boldsymbol{f}$ acting on a line passing through C_1 together with a couple $\boldsymbol{n}$. These forces impart an angular acceleration of $\dot{\boldsymbol{\omega}}_1$ to B_1 and a linear acceleration of $\boldsymbol{a}_1$ to its centre of mass. The problem is to express $\boldsymbol{a}_1$ and $\dot{\boldsymbol{\omega}}_1$ in terms of $\boldsymbol{f}$ and $\boldsymbol{n}$.

Diagram



Solution

The key to solving a problem like this is to realise that the joint introduces one degree of motion freedom between the two bodies, but also imposes one constraint on the forces that can be transmitted through the joint. The latter can be used to eliminate the former, at which point it becomes possible to express every force and acceleration in the system as a function of $\dot{\boldsymbol{\omega}}_1$ and $\boldsymbol{a}_1$. The problem is then solved by expressing $\boldsymbol{f}$ and $\boldsymbol{n}$ as functions of $\dot{\boldsymbol{\omega}}_1$ and $\boldsymbol{a}_1$, and then inverting the equations to express the accelerations in terms of the forces.

Let us introduce the following quantities. Let $\boldsymbol{f}_1$, $\boldsymbol{n}_1$, $\boldsymbol{f}_2$ and $\boldsymbol{n}_2$ be the net forces and couples acting on B_1 and B_2, respectively, where the lines of action of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ pass through C_1 and C_2, respectively; let $\dot{\boldsymbol{\omega}}_2$ and $\boldsymbol{a}_2$ be the angular acceleration of B_2 and the linear acceleration of its centre of mass; let ${}^P\boldsymbol{a}_1$, ${}^P\dot{\boldsymbol{\omega}}_1$, ${}^P\boldsymbol{a}_2$ and ${}^P\dot{\boldsymbol{\omega}}_2$ be the linear and angular accelerations of B_1 and B_2 expressed at P; and let ${}^P\boldsymbol{f}_2$, ${}^P\boldsymbol{n}_2$, ${}^1\boldsymbol{f}_2$ and ${}^1\boldsymbol{n}_2$ be the net force and couple acting on B_2 expressed at P and C_1, respectively. As the system of applied forces acts only on B_1, the net force and couple acting on B_2 are also the net force and couple transmitted through the joint. Let us also define $\boldsymbol{r}_1 = \overrightarrow{C_1P}$ and $\boldsymbol{r}_2 = \overrightarrow{C_2P}$, and let α be the joint acceleration variable.

The equations of motion of the two bodies, expressed at their centres of mass, are

$$\boldsymbol{f}_1 = m_1\,\boldsymbol{a}_1, \tag{1}$$

$$\boldsymbol{n}_1 = \boldsymbol{I}_1\,\dot{\boldsymbol{\omega}}_1, \tag{2}$$

$$\boldsymbol{f}_2 = m_2\,\boldsymbol{a}_2 \tag{3}$$

and

$$\boldsymbol{n}_2 = \boldsymbol{I}_2\,\dot{\boldsymbol{\omega}}_2. \tag{4}$$

(There are no velocity terms because the bodies are at rest.)

The rules for transferring forces and accelerations (of bodies at rest) from one point to another provide us with the following relationships between quantities referred to C_1, C_2 and P:

$${}^P\boldsymbol{a}_1 = \boldsymbol{a}_1 - \boldsymbol{r}_1 \times \dot{\boldsymbol{\omega}}_1, \tag{5}$$

$${}^P\boldsymbol{a}_2 = \boldsymbol{a}_2 - \boldsymbol{r}_2 \times \dot{\boldsymbol{\omega}}_2, \tag{6}$$

$${}^P\dot{\boldsymbol{\omega}}_1 = \dot{\boldsymbol{\omega}}_1, \tag{7}$$

$${}^P\dot{\boldsymbol{\omega}}_2 = \dot{\boldsymbol{\omega}}_2, \tag{8}$$

$${}^1\boldsymbol{f}_2 = {}^P\boldsymbol{f}_2 = \boldsymbol{f}_2, \tag{9}$$

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$$ {}^{P}\boldsymbol{n}_2 = \boldsymbol{n}_2 - \boldsymbol{r}_2 \times \boldsymbol{f}_2 \tag{10} $$

and

$$ {}^{1}\boldsymbol{n}_2 = \boldsymbol{n}_2 + (\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{f}_2 . \tag{11} $$

If B_1 exerts ${}^1\boldsymbol{f}_2$ and ${}^1\boldsymbol{n}_2$ on B_2 then B_2 exerts $-{}^1\boldsymbol{f}_2$ and $-{}^1\boldsymbol{n}_2$ on B_1 (Newton's 3rd law expressed at C_1); so the net force and couple acting on B_1 are

$$ \begin{aligned} \boldsymbol{f}_1 &= \boldsymbol{f} - {}^1\boldsymbol{f}_2 \\ \boldsymbol{n}_1 &= \boldsymbol{n} - {}^1\boldsymbol{n}_2, \end{aligned} $$

from which we get (via Eqs. 9 and 11)

$$ \boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2 \tag{12} $$

and

$$ \boldsymbol{n} = \boldsymbol{n}_1 + \boldsymbol{n}_2 + (\boldsymbol{r}_1 - \boldsymbol{r}_2) \times \boldsymbol{f}_2 . \tag{13} $$

The joint allows B_2 one degree of motion freedom relative to B_1, and imposes one constraint on the couple transmitted from B_1 to B_2. Expressed at P, the equations are

$$ {}^{P}\boldsymbol{a}_2 = {}^{P}\boldsymbol{a}_1, \tag{14} $$

$$ {}^{P}\dot{\boldsymbol{\omega}}_2 = {}^{P}\dot{\boldsymbol{\omega}}_1 + \boldsymbol{s}\,\alpha \tag{15} $$

and

$$ \boldsymbol{s}^{\mathrm{T}}\, {}^{P}\boldsymbol{n}_2 = 0. \tag{16} $$

(There is no constraint on ${}^{P}\boldsymbol{f}_2$. Eq. 16 is sufficient to ensure that the force and couple transmitted by the joint perform no work in the direction of relative motion permitted by the joint.)

We are now ready to solve the problem. Let us start by calculating $\boldsymbol{a}_2$ and $\dot{\boldsymbol{\omega}}_2$ in terms of $\boldsymbol{a}_1$, $\dot{\boldsymbol{\omega}}_1$ and α. From Eqs. 8, 15 and 7 we have

$$ \begin{aligned} \dot{\boldsymbol{\omega}}_2 &= {}^{P}\dot{\boldsymbol{\omega}}_2 \\ &= {}^{P}\dot{\boldsymbol{\omega}}_1 + \boldsymbol{s}\,\alpha \\ &= \dot{\boldsymbol{\omega}}_1 + \boldsymbol{s}\,\alpha, \end{aligned} \tag{17} $$

and from Eqs. 6, 14, 17 and 5 we have

$$ \begin{aligned} \boldsymbol{a}_2 &= {}^{P}\boldsymbol{a}_2 + \boldsymbol{r}_2 \times \dot{\boldsymbol{\omega}}_2 \\ &= {}^{P}\boldsymbol{a}_1 + \boldsymbol{r}_2 \times (\dot{\boldsymbol{\omega}}_1 + \boldsymbol{s}\,\alpha) \\ &= \boldsymbol{a}_1 - \boldsymbol{r}_1 \times \dot{\boldsymbol{\omega}}_1 + \boldsymbol{r}_2 \times (\dot{\boldsymbol{\omega}}_1 + \boldsymbol{s}\,\alpha) \\ &= \boldsymbol{a}_1 + (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \dot{\boldsymbol{\omega}}_1 + \boldsymbol{r}_2 \times \boldsymbol{s}\,\alpha. \end{aligned} \tag{18} $$

Now let us calculate α. From Eqs. 16, 10, 3, 4, 17 and 18 we get

$$ \begin{aligned} 0 &= \boldsymbol{s}^{\mathrm{T}}\, {}^{P}\boldsymbol{n}_2 \\ &= \boldsymbol{s}^{\mathrm{T}}(\boldsymbol{n}_2 - \boldsymbol{r}_2 \times \boldsymbol{f}_2) \\ &= \boldsymbol{s}^{\mathrm{T}}(\boldsymbol{I}_2\,\dot{\boldsymbol{\omega}}_2 - m_2\,\boldsymbol{r}_2 \times \boldsymbol{a}_2) \\ &= \boldsymbol{s}^{\mathrm{T}}(\boldsymbol{I}_2\,(\dot{\boldsymbol{\omega}}_1 + \boldsymbol{s}\,\alpha) - m_2\,\boldsymbol{r}_2 \times \\ &\qquad (\boldsymbol{a}_1 + (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \dot{\boldsymbol{\omega}}_1 + \boldsymbol{r}_2 \times \boldsymbol{s}\,\alpha)). \end{aligned} $$

Collecting terms in α gives

$$ \begin{aligned} &\boldsymbol{s}^{\mathrm{T}}(\boldsymbol{I}_2\,\boldsymbol{s} - m_2\,\boldsymbol{r}_2 \times (\boldsymbol{r}_2 \times \boldsymbol{s}))\,\alpha + \boldsymbol{s}^{\mathrm{T}}(\boldsymbol{I}_2\,\dot{\boldsymbol{\omega}}_1 \\ &\qquad - m_2\,\boldsymbol{r}_2 \times (\boldsymbol{a}_1 + (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \dot{\boldsymbol{\omega}}_1)) = 0, \end{aligned} $$

hence

$$ \alpha = -\frac{\boldsymbol{s}^{\mathrm{T}}(\boldsymbol{I}_2\,\dot{\boldsymbol{\omega}}_1 - m_2\,\boldsymbol{r}_2 \times (\boldsymbol{a}_1 + (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \dot{\boldsymbol{\omega}}_1))}{\boldsymbol{s}^{\mathrm{T}}(\boldsymbol{I}_2\,\boldsymbol{s} - m_2\,\boldsymbol{r}_2 \times (\boldsymbol{r}_2 \times \boldsymbol{s}))} \tag{19} $$

This equation is only valid if the denominator is not equal to zero, so we must investigate the necessary conditions for it to be nonzero. This problem can be solved using the following trick. For any two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, the cross product $\boldsymbol{u} \times \boldsymbol{v}$ can be expressed in the form $\boldsymbol{u} \times \boldsymbol{v} = \tilde{\boldsymbol{u}}\,\boldsymbol{v}$, where $\tilde{\boldsymbol{u}}$ is the skew-symmetric matrix

$$ \tilde{\boldsymbol{u}} = \begin{bmatrix} 0 & -u_z & u_y \\ u_z & 0 & -u_x \\ -u_y & u_x & 0 \end{bmatrix} . $$

Using this trick, we can express the denominator in the form $\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}$ where

$$ \begin{aligned} \boldsymbol{J} &= \boldsymbol{I}_2 - m_2\,\tilde{\boldsymbol{r}}_2\,\tilde{\boldsymbol{r}}_2 \\ &= \boldsymbol{I}_2 + m_2\,\tilde{\boldsymbol{r}}_2^{\mathrm{T}}\,\tilde{\boldsymbol{r}}_2 . \end{aligned} \tag{20} $$

$\boldsymbol{J}$ is therefore the sum of an SPD matrix and an SPSD matrix, hence itself also SPD, so the denominator of Eq. 19 is guaranteed to be strictly greater than zero. Substituting Eq. 20 into Eq. 19 gives us the following simplified expression for α:

$$ \alpha = -\frac{\boldsymbol{s}^{\mathrm{T}}(\boldsymbol{J}\,\dot{\boldsymbol{\omega}}_1 - m_2\,\boldsymbol{r}_2 \times (\boldsymbol{a}_1 - \boldsymbol{r}_1 \times \dot{\boldsymbol{\omega}}_1))}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}} \tag{21} $$

The next step is to express $\boldsymbol{f}$ and $\boldsymbol{n}$ in terms of $\boldsymbol{a}_1$, $\dot{\boldsymbol{\omega}}_1$ and α, and then to eliminate α using Eq. 21. Let us start with $\boldsymbol{f}$. From Eqs. 12, 1, 3 and 18 we get

$$ \begin{aligned} \boldsymbol{f} &= \boldsymbol{f}_1 + \boldsymbol{f}_2 \\ &= m_1\,\boldsymbol{a}_1 + m_2\,\boldsymbol{a}_2 \\ &= m_1\,\boldsymbol{a}_1 + m_2(\boldsymbol{a}_1 + (\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \dot{\boldsymbol{\omega}}_1 + \boldsymbol{r}_2 \times \boldsymbol{s}\,\alpha) \\ &= (m_1 + m_2)\boldsymbol{a}_1 + m_2(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \dot{\boldsymbol{\omega}}_1 \\ &\qquad + m_2\,\boldsymbol{r}_2 \times \boldsymbol{s}\,\alpha. \end{aligned} $$

Elimitating α using Eq. 21 gives

$$ \begin{aligned} \boldsymbol{f} &= (m_1 + m_2)\boldsymbol{a}_1 + m_2(\boldsymbol{r}_2 - \boldsymbol{r}_1) \times \dot{\boldsymbol{\omega}}_1 - \\ &\quad m_2\,\frac{\boldsymbol{r}_2 \times \boldsymbol{s}\,\boldsymbol{s}^{\mathrm{T}}(\boldsymbol{J}\,\dot{\boldsymbol{\omega}}_1 - m_2\,\boldsymbol{r}_2 \times (\boldsymbol{a}_1 - \boldsymbol{r}_1 \times \dot{\boldsymbol{\omega}}_1))}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}} \, ; \end{aligned} $$

and collecting terms in $\boldsymbol{a}_1$ and $\dot{\boldsymbol{\omega}}_1$ gives

$$\begin{aligned}\boldsymbol{f} &= \left(m_1 + m_2 + m_2^2\,\frac{\tilde{\boldsymbol{r}}_2\,\boldsymbol{s}\,\boldsymbol{s}^{\mathrm{T}}\,\tilde{\boldsymbol{r}}_2}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}}\right)\boldsymbol{a}_1 + \\ &\left(m_2(\tilde{\boldsymbol{r}}_2 - \tilde{\boldsymbol{r}}_1) - m_2\,\frac{\tilde{\boldsymbol{r}}_2\,\boldsymbol{s}\,\boldsymbol{s}^{\mathrm{T}}(\boldsymbol{J} + m_2\,\tilde{\boldsymbol{r}}_2\,\tilde{\boldsymbol{r}}_1)}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}}\right)\dot{\boldsymbol{\omega}}_1.\end{aligned} \tag{22}$$

Repeating the procedure for $\boldsymbol{n}$, Eqs. 13, 2, 3, 4, 17 and 18 give

$$\begin{aligned}\boldsymbol{n} &= \boldsymbol{n}_1 + \boldsymbol{n}_2 + (\boldsymbol{r}_1 - \boldsymbol{r}_2)\times\boldsymbol{f}_2 \\ &= \boldsymbol{I}_1\,\dot{\boldsymbol{\omega}}_1 + \boldsymbol{I}_2\,\dot{\boldsymbol{\omega}}_2 + m_2(\boldsymbol{r}_1 - \boldsymbol{r}_2)\times\boldsymbol{a}_2 \\ &= \boldsymbol{I}_1\,\dot{\boldsymbol{\omega}}_1 + \boldsymbol{I}_2(\dot{\boldsymbol{\omega}}_1 + \boldsymbol{s}\,\alpha) + m_2(\boldsymbol{r}_1 - \boldsymbol{r}_2)\times \\ &\quad(\boldsymbol{a}_1 + (\boldsymbol{r}_2 - \boldsymbol{r}_1)\times\dot{\boldsymbol{\omega}}_1 + \boldsymbol{r}_2\times\boldsymbol{s}\,\alpha) \\ &= (\boldsymbol{I}_1 + \boldsymbol{I}_2 - m_2(\tilde{\boldsymbol{r}}_1 - \tilde{\boldsymbol{r}}_2)^2)\dot{\boldsymbol{\omega}}_1 \\ &\quad + m_2(\tilde{\boldsymbol{r}}_1 - \tilde{\boldsymbol{r}}_2)\boldsymbol{a}_1 + \boldsymbol{K}\boldsymbol{s}\,\alpha,\end{aligned} \tag{23}$$

where

$$\begin{aligned}\boldsymbol{K} &= \boldsymbol{I}_2 + m_2(\tilde{\boldsymbol{r}}_1 - \tilde{\boldsymbol{r}}_2)\tilde{\boldsymbol{r}}_2 \\ &= \boldsymbol{J} + m_2\,\tilde{\boldsymbol{r}}_1\,\tilde{\boldsymbol{r}}_2.\end{aligned} \tag{24}$$

Note that Eq. 21 can now be simplified to

$$\alpha = -\frac{\boldsymbol{s}^{\mathrm{T}}(\boldsymbol{K}^{\mathrm{T}}\dot{\boldsymbol{\omega}}_1 - m_2\,\tilde{\boldsymbol{r}}_2\,\boldsymbol{a}_1)}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}}. \tag{25}$$

Eliminating α from Eq. 23 using Eq. 25 gives

$$\begin{aligned}\boldsymbol{n} &= (\boldsymbol{I}_1 + \boldsymbol{I}_2 - m_2(\tilde{\boldsymbol{r}}_1 - \tilde{\boldsymbol{r}}_2)^2)\dot{\boldsymbol{\omega}}_1 + \\ &\quad m_2(\tilde{\boldsymbol{r}}_1 - \tilde{\boldsymbol{r}}_2)\boldsymbol{a}_1 - \frac{\boldsymbol{K}\boldsymbol{s}\,\boldsymbol{s}^{\mathrm{T}}(\boldsymbol{K}^{\mathrm{T}}\dot{\boldsymbol{\omega}}_1 - m_2\,\tilde{\boldsymbol{r}}_2\,\boldsymbol{a}_1)}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}},\end{aligned}$$

and collecting terms in $\dot{\boldsymbol{\omega}}_1$ and $\boldsymbol{a}_1$ gives

$$\begin{aligned}\boldsymbol{n} &= \left(\boldsymbol{I}_1 + \boldsymbol{I}_2 - m_2(\tilde{\boldsymbol{r}}_1 - \tilde{\boldsymbol{r}}_2)^2 - \frac{\boldsymbol{K}\boldsymbol{s}\,\boldsymbol{s}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}}\right)\dot{\boldsymbol{\omega}}_1 \\ &\quad + \left(m_2(\tilde{\boldsymbol{r}}_1 - \tilde{\boldsymbol{r}}_2) + m_2\,\frac{\boldsymbol{K}\boldsymbol{s}\,\boldsymbol{s}^{\mathrm{T}}\tilde{\boldsymbol{r}}_2}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}}\right)\boldsymbol{a}_1.\end{aligned} \tag{26}$$

The final step is to combine Eqs. 22 and 26 into a single equation:

$$\begin{bmatrix}\boldsymbol{f} \\ \boldsymbol{n}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D}\end{bmatrix}\begin{bmatrix}\boldsymbol{a}_1 \\ \dot{\boldsymbol{\omega}}_1\end{bmatrix}, \tag{27}$$

where

$$\boldsymbol{A} = (m_1 + m_2)\,\mathbf{1}_{3\times 3} + m_2^2\,\frac{\tilde{\boldsymbol{r}}_2\,\boldsymbol{s}\,\boldsymbol{s}^{\mathrm{T}}\,\tilde{\boldsymbol{r}}_2}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}}, \tag{28}$$

$$\boldsymbol{B} = m_2(\tilde{\boldsymbol{r}}_2 - \tilde{\boldsymbol{r}}_1) - m_2\,\frac{\tilde{\boldsymbol{r}}_2\,\boldsymbol{s}\,\boldsymbol{s}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}}, \tag{29}$$

$$\boldsymbol{C} = m_2(\tilde{\boldsymbol{r}}_1 - \tilde{\boldsymbol{r}}_2) + m_2\,\frac{\boldsymbol{K}\boldsymbol{s}\,\boldsymbol{s}^{\mathrm{T}}\tilde{\boldsymbol{r}}_2}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}} \tag{30}$$

and

$$\boldsymbol{D} = \boldsymbol{I}_1 + \boldsymbol{I}_2 - m_2(\tilde{\boldsymbol{r}}_1 - \tilde{\boldsymbol{r}}_2)^2 - \frac{\boldsymbol{K}\boldsymbol{s}\,\boldsymbol{s}^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}}{\boldsymbol{s}^{\mathrm{T}}\boldsymbol{J}\,\boldsymbol{s}}. \tag{31}$$

($\mathbf{1}_{3\times 3}$ is an identity matrix.) Notice that $\boldsymbol{A}$ and $\boldsymbol{D}$ are symmetric, and that $\boldsymbol{B} = \boldsymbol{C}^{\mathrm{T}}$. The solution to the original problem is then

$$\begin{bmatrix}\boldsymbol{a}_1 \\ \dot{\boldsymbol{\omega}}_1\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D}\end{bmatrix}^{-1}\begin{bmatrix}\boldsymbol{f} \\ \boldsymbol{n}\end{bmatrix}. \tag{32}$$

At this point, we should prove that the 6×6 coefficient matrix is nonsingular. It is in fact an SPD matrix, but the easiest way to prove it is to show that it is identical to the solution obtained using the 6-D vector approach, which is easily shown to be an SPD matrix.