Spatial Vector Algebra

The Easy Way to do Rigid Body Dynamics

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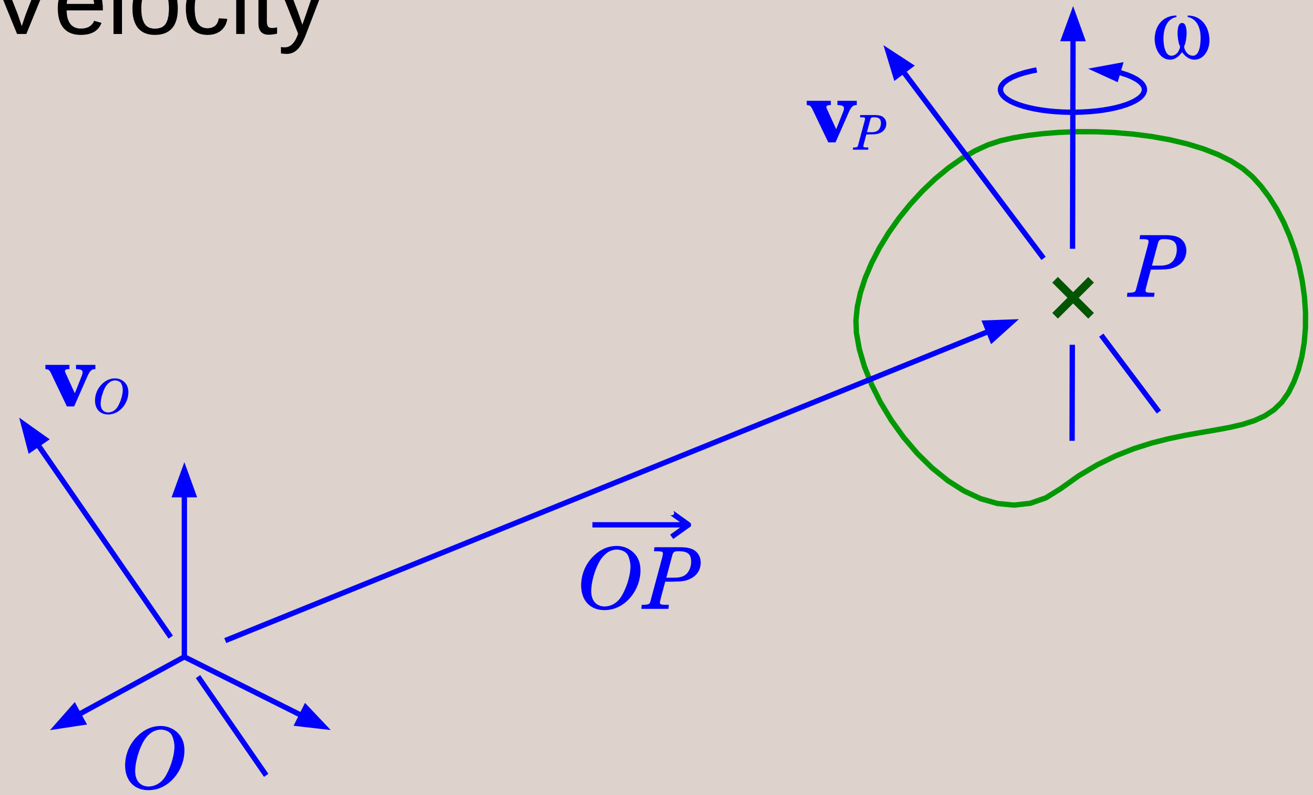
A concise vector notation for describing rigid−body velocity, acceleration, inertia, etc., using 6D vectors and tensors.

- fewer quantities
- fewer equations
- less effort
- fewer mistakes

Velocity



Spatial velocity: $\hat{\mathbf{v}} = \begin{bmatrix} \boldsymbol{\omega} \\ \mathbf{v}_O \end{bmatrix}$

$$(\mathbf{v}_O = \mathbf{v}_P + \overrightarrow{OP} \times \boldsymbol{\omega})$$

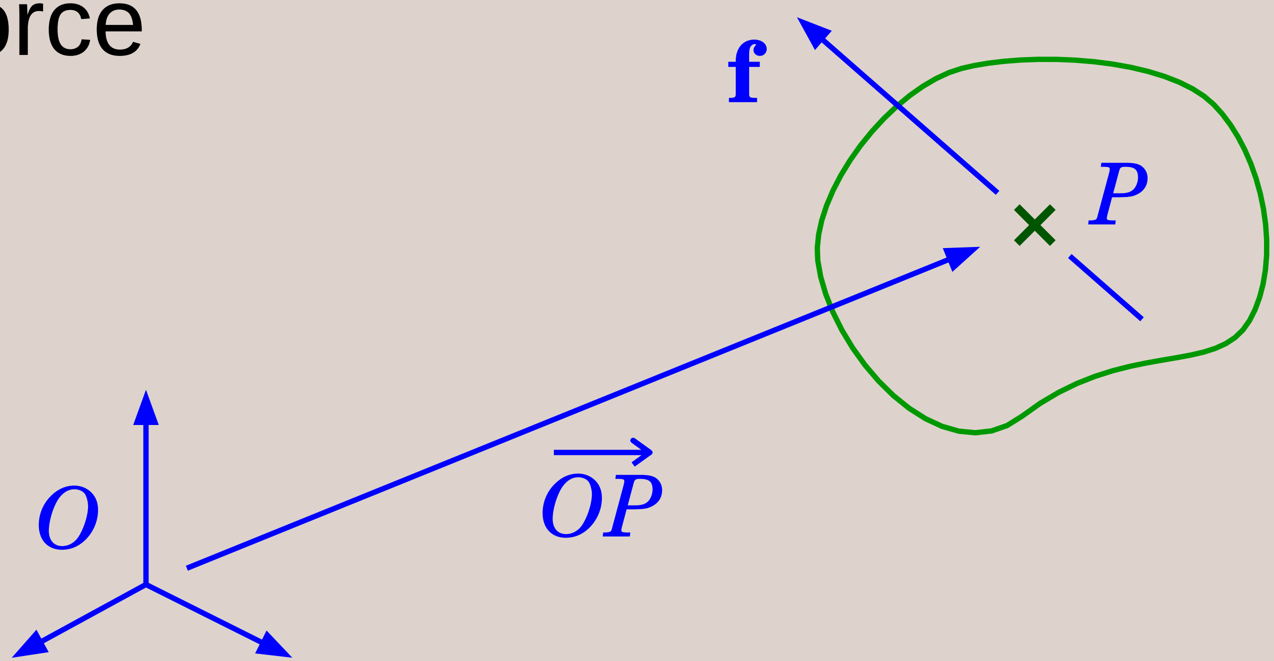
Acceleration

. . . is the rate of change of velocity:

$$\hat{\mathbf{a}} = \frac{d}{dt}\hat{\mathbf{v}} = \begin{bmatrix} \dot{\omega} \\ \dot{\mathbf{v}}_O \end{bmatrix}$$

but this is *not* the linear acceleration of any point in the body!

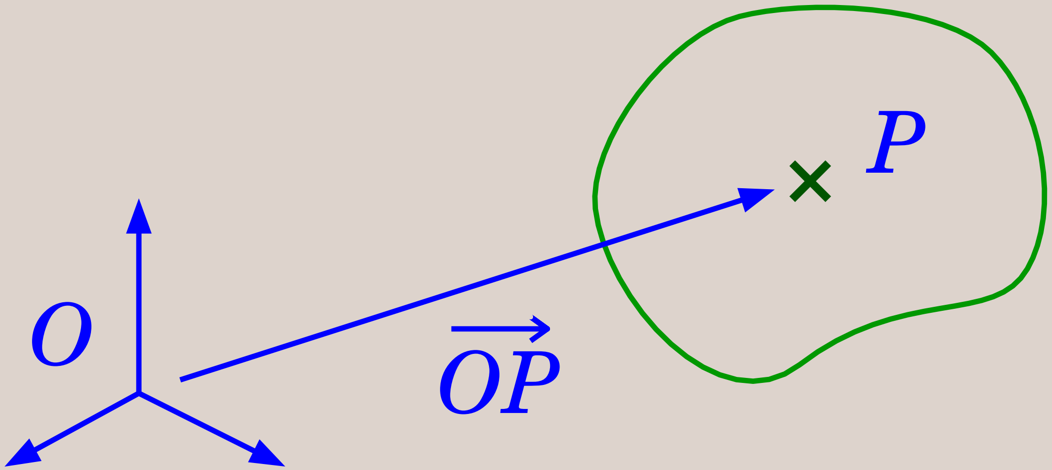
Force



force $\mathbf{f}$ through P: $$\hat{\mathbf{f}} = \begin{bmatrix} \overrightarrow{OP} \times \mathbf{f} \\ \mathbf{f} \end{bmatrix}$$

pure couple $\boldsymbol{\tau}$: $$\hat{\mathbf{f}} = \begin{bmatrix} \boldsymbol{\tau} \\ \mathbf{0} \end{bmatrix}$$

Rigid Body Inertia



mass: m

CoM: P

inertia at CoM: $\mathbf{I}^*$

spatial inertia tensor: $$\hat{\mathbf{I}} = \begin{bmatrix} \mathbf{I} & \mathbf{H} \\ \mathbf{H}^T & \mathbf{M} \end{bmatrix}$$

where $$\mathbf{M} = m\,\mathbf{1}$$
$$\mathbf{H} = m\,\overrightarrow{OP}\times$$
$$\mathbf{I} = \mathbf{I}^* - m\,\overrightarrow{OP}\times\,\overrightarrow{OP}\times$$

Operations on Spatial Quantities

- Composition of velocities

If $\hat{\mathbf{v}}_A$ = velocity of body A

$\hat{\mathbf{v}}_B$ = velocity of body B

$\hat{\mathbf{v}}_{BA}$ = relative velocity of B w.r.t. A

Then $\hat{\mathbf{v}}_B = \hat{\mathbf{v}}_A + \hat{\mathbf{v}}_{BA}$

- Composition of accelerations

If $\hat{\mathbf{a}}_A$ = acceleration of body A

$\hat{\mathbf{a}}_B$ = acceleration of body B

$\hat{\mathbf{a}}_{BA}$ = acceleration of B w.r.t. A

Then $\hat{\mathbf{a}}_B = \hat{\mathbf{a}}_A + \hat{\mathbf{a}}_{BA}$

Look, no Coriolis term!

- Composition of forces

If forces $\hat{\mathbf{f}}_1$ and $\hat{\mathbf{f}}_2$ both act on the same body then their resultant is

$$\hat{\mathbf{f}}_{tot} = \hat{\mathbf{f}}_1 + \hat{\mathbf{f}}_2$$

- Composition of inertias

If two bodies with inertias $\hat{\mathbf{I}}_A$ and $\hat{\mathbf{I}}_B$ are connected together then the inertia of the composite body is

$$\hat{\mathbf{I}}_{tot} = \hat{\mathbf{I}}_A + \hat{\mathbf{I}}_B$$

Mathematical Structure

spatial vectors inhabit *two* vector spaces:

M^6 — motion vectors

F^6 — force vectors

with a scalar product defined *between* them

$$\mathbf{m} \cdot \mathbf{f} = \mathit{work}$$

"$\cdot$" : $\mathsf{M}^6 \times \mathsf{F}^6 \mapsto \mathsf{R}$

Bases

If $\{\mathbf{d}_1, \ldots, \mathbf{d}_6\}$ is an arbitrary basis on M^6 then there exists a unique basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_6\}$ on F^6 satisfying

$$\mathbf{d}_i \cdot \mathbf{e}_j = \begin{cases} 0 : & i \neq j \\ 1 : & i = j \end{cases}$$

In this basis, the scalar product of two coordinate vectors is

$$\mathbf{m} \cdot \mathbf{f} = [\mathbf{m}]^T [\mathbf{f}]$$

Plücker Coordinates

A Cartesian coordinate frame $Oxyz$ defines *twelve* basis vectors:

$\mathbf{d}_{Ox}$, $\mathbf{d}_{Oy}$, $\mathbf{d}_{Oz}$, $\mathbf{d}_x$, $\mathbf{d}_y$, $\mathbf{d}_z$:

rotations about the Ox, Oy and Oz axes,

translations in the x, y and z directions

$\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$, $\mathbf{e}_{Ox}$, $\mathbf{e}_{Oy}$, $\mathbf{e}_{Oz}$:

couples in the yz, zx and xy planes, and

forces along the Ox, Oy and Oz axes

Equations like $\hat{\mathbf{v}} = \begin{bmatrix} \omega \\ \mathbf{v}_O \end{bmatrix}$ and $\hat{\mathbf{f}} = \begin{bmatrix} \tau_O \\ \mathbf{f} \end{bmatrix}$

really mean

$$\hat{\mathbf{v}} = \omega_x\, \mathbf{d}_{Ox} + \omega_y\, \mathbf{d}_{Oy} + \omega_z\, \mathbf{d}_{Oz} + \\ + v_{Ox}\, \mathbf{d}_x + v_{Oy}\, \mathbf{d}_y + v_{Oz}\, \mathbf{d}_z$$

$$\hat{\mathbf{f}} = \tau_{Ox}\, \mathbf{e}_x + \tau_{Oy}\, \mathbf{e}_y + \tau_{Oz}\, \mathbf{e}_z + \\ + f_x\, \mathbf{e}_{Ox} + f_y\, \mathbf{e}_{Oy} + f_z\, \mathbf{e}_{Oz}$$

Equation of Motion

$$\mathbf{f} = \frac{d}{dt}(\mathbf{I}\,\mathbf{v}) = \mathbf{I}\,\mathbf{a} + \mathbf{v} \times \mathbf{I}\,\mathbf{v}$$

$\mathbf{f}$ = net force acting on a rigid body

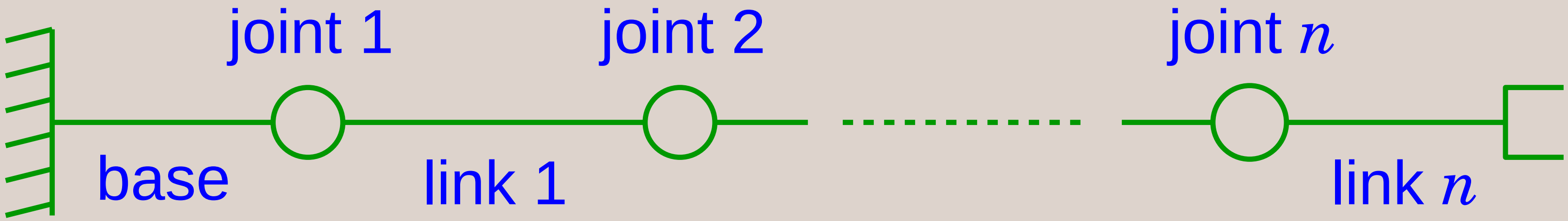
$\mathbf{I}$ = inertia of rigid body

$\mathbf{v}$ = velocity of rigid body

$\mathbf{I}\,\mathbf{v}$ = momentum of rigid body

$\mathbf{a}$ = acceleration of rigid body

Example 1: Robot Kinematics



$$\mathbf{v}_i = \mathbf{v}_{i-1} + \mathbf{s}_i\,\dot{q}_i \qquad (\mathbf{v}_0 = \mathbf{0})$$

$$\mathbf{a}_i = \mathbf{a}_{i-1} + \dot{\mathbf{s}}_i\,\dot{q}_i + \mathbf{s}_i\,\ddot{q}_i \qquad (\mathbf{a}_0 = \mathbf{0})$$

$\mathbf{v}_i,\ \mathbf{a}_i$ link velocity and acceleration

$\dot{q}_i,\ \ddot{q}_i,\ \mathbf{s}_i$ joint velocity, acceleration & axis

Example 2: Inverse Dynamics

(Calculate the joint torques Q_i that will produce the desired joint accelerations $\ddot{q}_i$.)

$$\mathbf{v}_i = \mathbf{v}_{i-1} + \mathbf{s}_i\,\dot{q}_i \qquad (\mathbf{v}_0 = \mathbf{0})$$

$$\mathbf{a}_i = \mathbf{a}_{i-1} + \dot{\mathbf{s}}_i\,\dot{q}_i + \mathbf{s}_i\,\ddot{q}_i \qquad (\mathbf{a}_0 = \mathbf{0})$$

$$\mathbf{f}_i = \mathbf{f}_{i+1} + \mathbf{I}_i\,\mathbf{a}_i + \mathbf{v}_i \times \mathbf{I}_i\,\mathbf{v}_i \qquad (\mathbf{f}_{n+1} = \mathbf{f}_{ee})$$

$$Q_i = \mathbf{s}_i^T\,\mathbf{f}_i$$

(The Recursive Newton−Euler Algorithm)